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PLANE AND SOLID GEOMETRY.

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THE

ELEMENTS

OF

PLANE AND SOLID GEOMETRY;

WITH CHAPTERS ON

MENSURATION

AND

MODERN GEOMETRY.

BY

ISAAC SHARPLESS,

PROFESSOR OF MATHEMATICS IN HAVERFORD COLLEGE, PA.

PORTER & COATES,
PHILADELPHIA.

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NOTE.

THROUGH its various stages this work has had the benefit of the suggestions and criticisms of Thomas K. Brown, Mathematical Teacher of Westtown School, whose Algebra has been prepared simultaneously with it. The authors have kept in view the object of making these books parts of a connected series of Mathematical text-books.

The following volumes are now completed:

ELEMENTARY ALGEBRA. By THOMAS K. BROWN.
ELEMENTARY PLANE GEOMETRY. By ISAAC SHARPLESS.
ELEMENTS OF PLANE AND SOLID GEOMETRY. By ISAAC SHARPLESS.

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PREFACE.

THIS little work on elementary Geometry is offered to those who desire to make themselves familiar with the general principles of the science in the limited time usually allotted to it. Beginning with simple definitions, and with postulates and axioms obvious to every one, it develops, in an unbroken series of propositions, the essential truths of Geometry. It differs alike from those treatises whose main object is to present the subject in its shortest and simplest form, and from those which are exhaustive and comprehensive in their scope.

The student does not meet at the outset, as in most modern Geometries, a collection of theorems involving constructions which he has not been taught to perform, but a simple problem; and, as he needs them, he finds other problems, so that no figure is called for which he has not the means to construct accurately and intelligently. He is impressed with the logical idea that correct conclusions can only be deduced from known premises, and he acquires practical skill in construction by applying the problems to every proposition.

While the basis of elementary Geometry is, and must ever be, contained in the works of Euclid, modern geometers, especially in France, have made important additions and corrections. Euclid's methods are sometimes cumbersome, and his omissions, especially in solid Geometry, are serious; yet years of experience attest the beneficial results of his teaching. His students think accurately and scientifically, and their training shows itself in their future work.

It has been an aim, in the preparation of this treatise, to incorporate with these advantages the improvements and additions which recent study has suggested. While thus increasing the scope of the work, its size has been kept down by such an arrangement of the problems and theorems as secures the simplest demonstrations.

PART I. treats of Plane Geometry. The order of propositions here adopted, seems to accomplish, as fully as possible, the two ends of keeping practice always in advance of theory, and removing difficulties, as much as strict logic will permit, from the path of the beginner. These ends gained, the introduction to the science becomes interesting and suggestive. Analyses of the proof, showing at a glance the relations of the different parts of the demonstration, are given at the close of some of the propositions; thus the student has the choice of two statements, which will assist each other in giving a correct understanding of the methods. At the end of each book is a collection of exercises for original investigation and practical application. They are so explained that the average student should be able to solve them unaided—not so difficult as to cause him to give up in despair, nor so easy as to be of no interest or value. A collection of rules for the mensuration of plane surfaces, and a number of examples in them, complete Part I. This part is bound separately for the convenience of those who desire such a limited course.

PART II. contains three books, treating respectively of the geometry of planes, of solids, and of spherical geometry, with accompanying exercises. The rules for the measurement of geometrical solids are collected; a few additional rules in mensuration, not previously referred to, are proved, and numerous examples given.

PART III. contains an introduction to *Modern Geometry*, the name usually given to the discoveries in pure geometrical science, made since the advance in this direction was stayed, for a time, by the brilliant prospects opened by the Analytical Geometry of DESCARTES. For much of this part the author is indebted to the TRAITÉ DE GÉOMÉTRIE of ROUCHÉ ET DE COMBEROUSSE.

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INTRODUCTION.

EXPLANATION OF TERMS.

THE study of form is the basis of Geometry. A cubic foot of matter may be in the shape of a ball, or of a cubical block, or it may be irregular. When thus regarding only the amount of material, no attention is paid to the outline; Geometry, however, considers the outline to the exclusion of the amount of matter which it encloses.

A geometrical sphere is not a sphere of iron or wood, but a sphere of empty space. It is therefore an imaginary solid, which cannot be perceived by the senses, and for which we must use some representative, as a ball or a diagram, in order to describe it. It is a type of one class of geometrical magnitudes—**solids.**

These ideal portions of space are bounded by **surfaces** without thickness. Here, again, Geometry deals with the *form* of the surface. The surface may be flat, so that a straight-edged ruler, in whatever direction it be laid, will touch along its length, or it may be curved. It may, if flat, be limited by straight lines or curved, by lines of equal or unequal length, etc.

If, now, we suppose the edge of the ruler to be without breadth, we obtain an idea of a new kind of magnitude—a geometrical **line.** As solids are bounded by surfaces, so surfaces are bounded by lines. Lines are also imaginary, having neither breadth nor thickness. We use marks to represent them in our books. The *form* of the line is again important, whether it be straight or curved, long or short.

These remarks prepare the way for the more concise definitions which follow.

1. **Geometry** is that science which treats of the properties, relations and measurement of magnitudes.

Any solid may be considered as having three dimensions—length, breadth and thickness. A **geometrical solid** is the portion of space enclosed within the boundaries of a physical solid. These boundaries are surfaces; the boundaries of surfaces are lines, and lines are limited by points. The term **magnitudes** applies to solids, surfaces and lines. A **solid** has extension in three directions, a **surface** in two, and a **line** in one. A **point** has no magnitude.

2. A **proposition** is a general statement. It may be a theorem, problem, axiom or postulate.

3. A **theorem** is a statement which it is required to prove.

4. A **problem** is a question which it is required to solve.

5. A **demonstration** is the course of reasoning by which the theorem is proved or problem solved.

A theorem may be proved either directly or by showing that an absurdity would result if it were supposed untrue.

6. An **axiom** is a self-evident proposition requiring no proof.

Such as: The whole is greater than a part.

7. A **postulate** is something to be done which is so simple that no one will hesitate to allow it.

Such as: Two points may be joined by a straight line.

8. A **corollary** is a consequence drawn from a preceding proposition.

9. A **scholium** is a remark made upon a preceding proposition, showing its limits or application.

10. An **hypothesis** is a proposition assumed to be true in order to argue from.

The following expressions show the meaning of signs used through the book:

$A = B$	means that	A is equal to B.
$A > B$	"	A is greater than B.
$A < B$	"	A is less than B.
$A + B$	"	A is added to B.
$A - B$	"	B is subtracted from A.
$A \times B$, $A . B$	"	A is multiplied by B.
$\frac{A}{B}$	"	A is divided by B.
$2A$	"	A is taken twice, or $A + A$.
A^2	"	A is taken twice as a factor, or $A \times A$.

$\triangle ABC$ means the triangle ABC, as distinguished from the angle ABC.

$\therefore$ means therefore.

Figures in parenthesis through a demonstration, thus, (I. 15), (VI. 2), refer to the previous proposition in which the statement was proved, meaning (Book I. Prop. 15), (Book VI. Prop. 2).

(*Hyp.*)	stands for	Hypothesis.
(*Ax.*)	"	Axiom.
(*Post.*)	"	Postulate.
(*Cor.*)	"	Corollary.
(*Sch.*)	"	Scholium.
(*Constr.*)	"	Construction.

PART I.

PLANE GEOMETRY.

BOOK I.

LINES, ANGLES, TRIANGLES, PARALLELOGRAMS.

DEFINITIONS.

Point.—Line.—Surface.

1. **A point** is that which has position, but not magnitude.

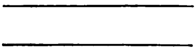
2. A **line** is length without breadth.

The extremities of lines are points. The intersection of one line with another is also a point.

3. If a line preserve the same direction throughout, it is a *straight line.* If it change its direction at every point, it is a *curved line* or *curve.*

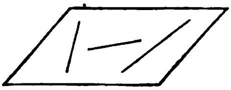
Corollary. Two straight lines cannot enclose space; neither can they coincide in any two points without coinciding altogether.

4. **Parallel straight lines** are such as are in the same plane, and being produced ever so far both ways do not meet.



5. A **surface** is that which has length and breadth only.

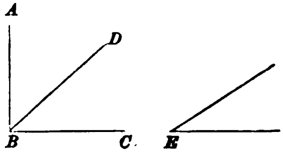
6. A **plane surface** is one in which, if any two points be taken, the straight line between them lies wholly in that surface.



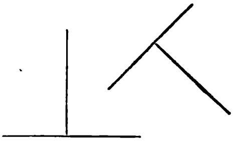
Angle.

7. A **plane rectilineal angle** is the inclination to each other of two straight lines which meet.

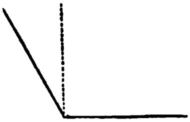
If a number of angles be at one point, as *B*, they are designated by three letters, of which the middle one is at the vertex of the angle. Thus, *DBC* means the angle formed by the two lines *DB*, *BC*. If there be but one angle at a point, as at *E*, it may be designated simply as the angle *E*.



8. When a straight line standing on another straight line makes the adjacent angles equal to each other, each of them is called a *right angle*, and the straight line which stands on the other is called a *perpendicular*.



9. An **obtuse**[1] **angle** is one that is greater than a right angle.



10. An **acute**[2] **angle** is one that is less than a right angle.

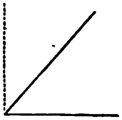


Figure.

11. A **figure** is that which is enclosed by one or more boundaries.

12. **Rectilineal**[3] **figures** are those contained by straight lines;

13. **Triangles**[4] by three straight lines;

---

[1] *Obtusus*, blunt.

[2] *Acutus*, sharp-pointed.

[3] *Rectus*, right; *linea*, a line.

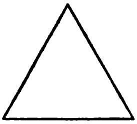
[4] *Tres, tria*, three; *angulus*, a corner.

14. **Quadrilaterals**[1] by four straight lines;

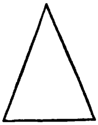
15. **Polygons**[2] by any number of straight lines.

Triangle.

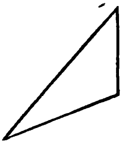
16. Considering only the sides of a triangle, an **equilateral**[3] **triangle** is one which has three equal sides.



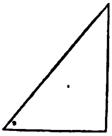
17. An **isosceles**[4] **triangle** is one which has two equal sides.



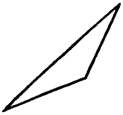
18. A **scalene**[5] **triangle** is one which has three unequal sides.



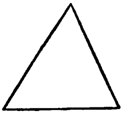
19. Considering only the angles of a triangle, a **right-angled triangle** is one which contains a right angle.



20. An **obtuse-angled triangle** is one which contains an obtuse angle.



21. An **acute-angled triangle** is one which contains three acute angles.



Quadrilateral.

22. Of quadrilaterals, a **parallelogram** is one which has the opposite sides parallel to each other.

---

[1] *Quatuor*, four; *latus*, a side.

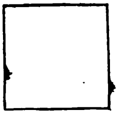
[2] *Poly*, many; *gōnia*, an angle.

[3] *Equus*, equal, *latus*, a side.

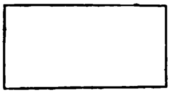
[4] *Isos*, equal; *scelos*, a leg.

[5] *Scalenus*, unequal.

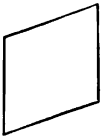
23. A **square** is one which has all its sides equal, and all its angles right angles.



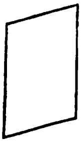
24. A **rectangle**[1] is one which has its opposite sides equal and all its angles right angles.



25. A **rhombus** is one which has all its sides equal, but its angles not right angles.

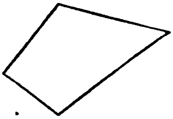


26. A **rhomboid** is one which has its opposite sides equal, but all its sides are not equal, nor its angles right angles.

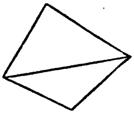


27. A **trapezoid** is one which has one pair of opposite sides parallel.

28. A **trapezium** is one which has no two sides parallel.

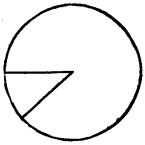


29. The **diagonal**[2] of a quadrilateral is a straight line joining its opposite angles.



Circle.

30. A **circle** is a plane figure, enclosed by one line, which is called the *circumference*, and is such that all lines drawn from a certain point within the figure to the circumference are equal to one another.



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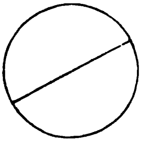
[1] *Rectus*, right; *angulus*, an angle.

[2] *Dia*, through; *gōnia*, an angle.

31. These lines are called *radii*, and this point is the *centre* of the circle.

32. A **diameter** of a circle is the straight line drawn through the centre, and terminated both ways by the circumference.

33. A **semicircle** is the figure contained by the diameter and the part of the circumference cut off by it.



34. The **perimeter** of a polygon is the sum of the lines which bound it. The perimeter of a circle is its circumference.

Postulates.

1. Let it be granted that a straight line may be drawn from any point to any other point;

2. That a terminated straight line may be produced to any length in a straight line;

3. That a circle may be described from any centre, with a radius equal to any given line.

Axioms.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals, the wholes are equal.

3. If equals be subtracted from equals, the remainders are equal.

4. If equals be added to unequals, the wholes are unequal.

5. If equals be subtracted from unequals, the remainders are unequal.

6. Things which are doubles of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another are equal to one another.

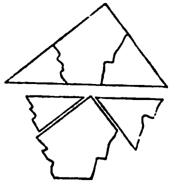
9. The whole is greater than its part.

10. All right angles are equal to one another.

11. Two straight lines which intersect each other cannot both be parallel to the same straight line.

12. The shortest distance between two points is the straight line which joins them.

NOTE.—The student should carefully construct all the figures he uses in the following propositions, drawing perpendiculars, equal angles, squares, etc. accurately with ruler and compass, and by the methods described in their proper places. He may also impress upon his mind the truth of some of the theorems by actual experiment. To illustrate: The 30th Prop. of Book I. proves that the three angles of any triangle are equal to two right angles. Let him draw, on paper, triangles of various shapes, and cut them out; then cut off two of the angles, and place them contiguous to the third.



Analyses are given to some of the propositions of Book I. As the exercise is very useful, the student is advised to continue it through the work. After the examples given, he need have but little difficulty in applying analyses to most propositions; indeed, he will find it to lessen the labor of understanding and remembering the proofs, by giving him a correct and comprehensive idea of the method.

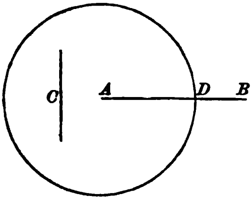
He should be careful, in studying a demonstration, to understand the reason of every step. Nothing is assumed except the definitions, postulates and axioms; and every statement may be referred to these, or to some previous proposition which itself is founded upon them. Taking them for granted, the whole geometry follows as a logical necessity. A thorough, honest mastery of the first half of the first book will make the remainder of geometry, and higher mathematics founded on geometry, interesting and comparatively easy.

The *method* usually followed in the propositions is—

1. A *general statement* or enunciation of the theorem to be proved or problem to be solved.
2. A *restatement* of it as applied to the particular figure on the paper.
3. The *construction* of any additional lines which may be needed to prove the proposition.
4. The *proof*, which terminates with the statement which we wished to prove.
5. Any *corollaries* or *scholiums* which may attach to the proposition.

Proposition 1.

***Problem.**—From the greater of two given straight lines, to cut off a part equal to the less.*



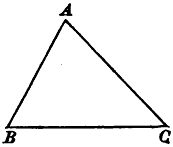
Restatement. Let AB and C be two straight lines, AB being the greater: it is required to cut off from AB a part equal to C.

Construction. With the centre A, and radius equal to C, describe (Post. 3) a circle, cutting AB in D.

Proof. AD is a radius of the circle, and is therefore equal (Def. 30) to C.

Proposition 2.

***Theorem.**—Two sides of a triangle are together greater than the third side.*



Restatement. Let ABC be a triangle; any two sides are together greater than the third.

Proof. Because (Ax. 12) the shortest distance between two points is the straight line which joins them, the line BC is the shortest distance from B to C, and is therefore less than BA and AC. Therefore BA and AC are together greater than BC. In the same way, any other two sides may be shown to be together greater than the third.

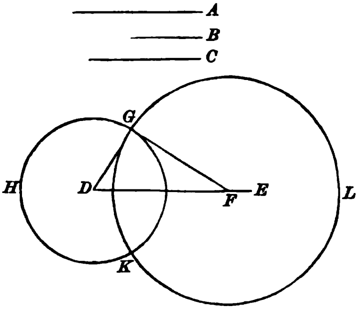
Proposition 3.

Problem.—*To construct a triangle, having given the three sides.*

Restatement. Let A, B, C be the three sides of a triangle: it is required to construct it.

It is a necessary condition that any two of the sides be greater than the third.

Construction. Lay down the indefinite line DE, and from it cut off (I. 1) DF equal to A. With D as a centre, and radius equal to B, describe (Post. 3) the circle GHK. With F as a centre, and radius equal to C, describe (Post. 3) the circle GKL. From G, where these circles cut, draw (Post. 1) the lines GD, GF. The triangle GDF is the required triangle.

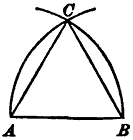


Proof. Because DG is a radius of the circle GHK, it is equal (Def. 30) to B; and because FG is a radius of GKL, it is equal to C; and DF was made equal to A; therefore GDF is a triangle having its three sides respectively equal to A, B, C.

Scholium.—In practice we need not describe the whole circle. Two small arcs intersecting at G will be sufficient.

Corollary.—If the triangle is to be equilateral (Def. 15), it is only necessary to know one side, *AB*.

From *A* and *B*, with radii equal to *AB*, describe (Post. 3) arcs of circles cutting in *C*. *ABC* is an equilateral triangle.

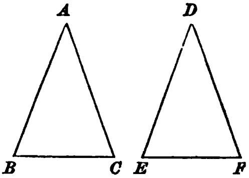


Proposition 4.

Theorem.—*If two triangles have the three sides of the one equal to three sides of the other, each to each, the angles of one will be equal to the angles of the other, each to each; the equal angles being opposite to the equal sides; and the areas of the triangles will be equal.*

Restatement. Let the triangles *ABC*, *DEF*, have their sides equal, each to each, *AB* to *DE*, *BC* to *EF*, and *AC* to *DF*; then will the angle *ABC* be equal to the angle *DEF*, *ACB* to *DFE*, and *BAC* to *EDF*; and the triangles *ABC*, *DEF* will be equal in area.

Proof. Let the triangle *ABC* be applied to the triangle *DEF*, so that *B* shall be on *E*, and *BC* on *EF*; and because *BC* is equal to *EF*, *C* will be on *F*. Also, if *A* will not fall on *D*, it will take a position either within the triangle *DEF*, on a side of *DEF* other than *D*, or outside of *DEF*. We will consider the three cases separately, and show they each lead to an impossibility.



Case 1. 1. Suppose the triangle *BAC* to take the position *EGF*, *A* falling at *G* inside the triangle *DEF*. Produce *EG* to meet *DF* in *H*.

Now (I. 2), $ED+DH>EH$.

Add *HF* to both; then (Ax. 4),

$$ED+DF>EH+HF.$$

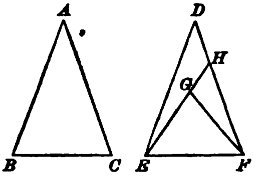
Also (I. 2), $FH + HG > FG$.

Add EG to both; then (Ax. 4),

$$EH + HF > EG + GF.$$

But $$ED + DF > EH + HF.$$

Still more, then, is $ED + DF > EG + GF$.



But because (Hyp.) the sides of BAC, that is, EGF, are equal to the sides of EDF,

$$ED + DF = EG + GF.$$

But it has been proved greater, which is absurd. Hence the supposition that A would fall inside of EDF is impossible.

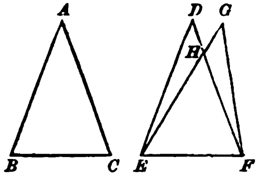
Case 2. 2. Suppose G to fall at H in last case. We have shown that

$$ED + DF > EH + HF.$$

But $$ED + DF = EH + HF,$$ which is impossible.

Case 3. 3. Suppose G to fall outside of DEF. Then (I. 2),

$$DH + HE > DE,$$



and (I. 2), $FH + HG > GF$.

Adding these expressions,

$$DF + GE > DE + GF.$$

But (Hyp.) $DF = GF$, and $GE = DE$.

$\therefore$ $$DF + GE = DE + GF.$$

Hence G will not fall outside DEF.

Therefore the triangle BAC would take no other position than EDF.

Hence the two triangles would coincide entirely, and be equal in all their parts (Ax. 8).

Scholium.—Triangles whose sides are equal, each to each, are said to be *mutually equilateral*, and triangles whose angles are equal, each to each, are *mutually equiangular*.

NOTE.—This is an example of an *indirect* demonstration. We show that all suppositions, contrary to the theorem, lead to contradictions. Hence, the theorem must be true. This method is sometimes called *reductio ad absurdum*.

Because we have proved that mutually equilateral triangles are mutually equiangular, it must not be supposed that the converse is true—viz., that mutually equiangular triangles are mutually equilateral. Two triangles may have their angles equal, each to each, and their sides unequal. In this case they will, however, be similar.

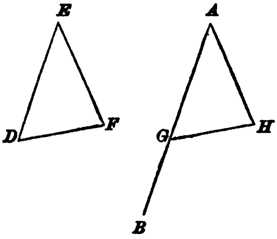
When a theorem is true, the converse is generally true, though not always; it is never safe to assume it without proof.

Proposition 5.

Problem.—At a given point in a straight line to make an angle equal to a given angle.

Restatement. Let A be a given point in the line AB, and DEF a given angle. It is required to make an angle at A equal to DEF.

Construction. Take any two points, D and F, in the sides of the angle, and join DF. Construct (I. 3) the triangle AGH, making $AG = ED$, $AH = EF$, and $GH = DF$.



Proof. Then because the two triangles EDF, AGH are mutually equilateral, the angles DEF, GAH, opposite the equal sides, are equal

(I. 4). Therefore at *A* the angle *BAH* has been constructed equal to *DEF*.

ANALYSIS.

By constr., $AG = DE$
" $AH = EF$ → (I. 4), $GAH = DEF$.
" $GH = DF$

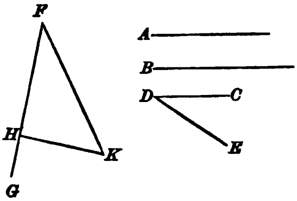
Proposition 6.

***Problem.**—To construct a triangle—*

1. Having two sides and the included angle given;

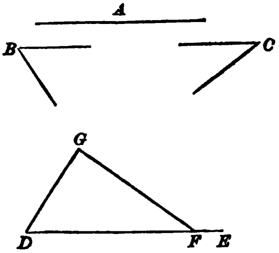
2. Having two angles and the included side given.

1. Let *A*, *B*, and *CDE* be two sides and the included angle of a triangle. It is required to construct it.



Lay down the indefinite line *FG*, and from it cut off (I. 1) *FH* equal to *A*. At *F* make (I. 5) the angle *HFK* equal to *CDE*, and cut off (I. 1) *FK* equal to *B*; join *HK*. Then the triangle *FHK* is the required triangle.

2. Let *A*, *B*, *C* be a side and the two adjacent angles of a triangle. It is required to construct it.



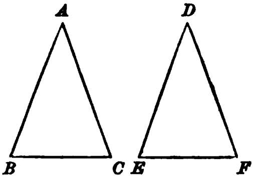
Lay down the indefinite line *DE*, and from it cut off (I. 1) *DF* equal to *A*. At *D* make (I. 5) the angle *EDG* equal to *B*; and at *F*, *DFG* equal to *C*. Let the lines *DG*, *FG* meet in *G*; then *DFG* is the required triangle.

Proposition 7.

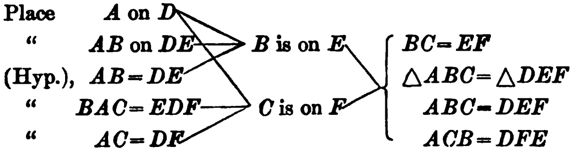
Theorem.*—If two triangles have two sides and the included angle of one, equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.*

Let *ABC*, *DEF* be two triangles which have *BA*, *AC*, and the angle *A*, equal to *ED*, *DF* and the angle *D*, each to each; then will the triangles be equal in all their parts.

Let the triangle *ABC* be applied to the triangle *DEF*, so that *A* shall be on *D*, and *AB* on *DE*. Then, because *AB* is equal to *DE*, *B* will coincide with *E*; also because the angle *BAC* is equal to the angle *EDF*, *AC* will coincide with *DF*, and because *AC* is equal to *DF*, *C* will coincide with *F*. Hence, *B* coinciding with *E*, and *C* with *F*, *BC* will coincide with *EF*, and the triangles will coincide, and are therefore equal (Ax. 8) in all their parts.



ANALYSIS.

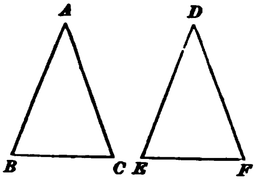


Proposition 8.

Theorem.*—If two triangles have two angles and the included side of one, equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.*

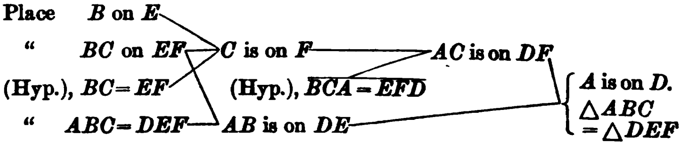
Let *ABC*, *DEF* be two triangles which have the two angles *ABC*, *ACB*, and the side *BC*, respectively equal to

DEF, *DFE*, and the side *EF*. The triangles will be equal in all their parts.



Apply the triangle *ABC* to the triangle *DEF*, so that *B* will be on *E*, and *BC* on *EF*; and because *BC* is equal to *EF*, *C* will coincide with *F*. Because the angle *ABC* is equal to the angle *DEF*, *BA* will coincide with *ED*; and because the angle *BCA* is equal to the angle *EFD*, *CA* will coincide with *FD*; hence *A* will coincide with *D*. Therefore the triangles will coincide altogether, and are equal in all their parts.

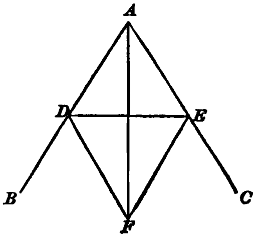
ANALYSIS.



Proposition 9.

Problem.—To bisect a given rectilineal angle.

Let *BAC* be a rectilineal angle. It is required to bisect it.



Take any point *D*, in *AB*, and from *AC* cut off (I. 1) *AE* equal to *AD*; join *DE*, and on it describe (I. 3, Cor.) the equilateral triangle *DFE*; join *AF*. *AF* will bisect *BAC*.

Because in the two triangles *DAF*, *EAF*, *AD* is equal to *AE*, *DF* to *EF*, and *AF* common, therefore (I. 4), the angle *DAF* is equal to the angle *EAF*. Therefore *BAC* is bisected by *AF*.

ANALYSIS.

$$\begin{tikzcd}[column sep=small]
\text{(By Constr.),} & AD=AE \arrow[dr] & \\
\text{(Def. 16),} & DF=EF \arrow[r] & \text{(I. 4), } DAF=EAF. \\
 & AF \text{ common} \arrow[ur] &
\end{tikzcd}$$

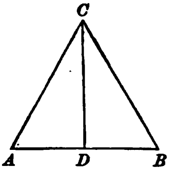
Scholium.—By the same construction, each of the halves *DAF*, *EAF* may be again bisected, and thus by successive bisections a given angle may be divided into 4, 8, 16, etc. equal parts.

Proposition 10.

Problem.—*To bisect a given finite straight line.*

Let *AB* be a straight line. It is required to bisect it.

Describe (I. 3, Cor.) on *AB* the equilateral triangle *ABC*, and bisect (I. 9) the angle *ACB* by the line *CD*, meeting *AB* in *D*; then will *AB* be bisected in *D*.



Because in the two triangles *ADC*, *BDC*, *AC* is equal to *CB*, *CD* is common, and the angle *ACD* equal to the angle *BCD*, therefore (I. 7) *AD* is equal to *BD*, and *AB* is bisected in *D*.

ANALYSIS.

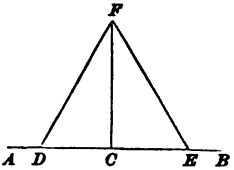
$$\begin{tikzcd}[column sep=small]
\text{(Def. 16),} & AC=BC \arrow[dr] & \\
 & CD \text{ common} \arrow[r] & \text{(I. 7), } AD=BD. \\
\text{(By Constr.),} & ACD=BCD \arrow[ur] &
\end{tikzcd}$$

Proposition 11.

Problem.—*To draw a straight line at right angles to a given straight line from a given point in it.*

Let *AB* be a straight line, and *C* a given point in it. It is required to draw from *C* a straight line at right angles to *AB*.

Take any point *D*, in *AC*, and cut off (I. 1) from *CB*, *CE* equal to *CD*. On *DE* describe (I. 3, Cor.) the equilateral triangle *DFE*; join *CF*. *CF* is at right angles to *AB*.



Because in the triangles *DCF*, *ECF*, *DC* is equal to *EC*, *DF* to *EF*, and *CF* common, therefore (I. 4) the angles *DCF*, *ECF*, opposite the equal sides, are equal. But when a straight line standing on another straight line makes the adjacent angles equal to each other, each of them is a right angle. (Def. 8.) Therefore *CF* is at right angles to *AB*.

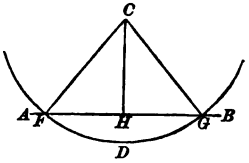
ANALYSIS.

$$\begin{tikzcd}
\text{(By Constr.),} & DC=CE \arrow[dr] & \\
\text{(Def. 16),} & DF=EF \arrow[r] & \text{(I. 4), } DCF=ECF. \\
& CF \text{ common} \arrow[ur] &
\end{tikzcd}$$

Proposition 12.

Problem.—*To draw a straight line perpendicular to a given straight line of unlimited length, from a given point without it.*

Let *AB* be a straight line, and *C* a point without it. It is required to draw from *C* a perpendicular to *AB*.



Take any point, *D*, on the opposite side of *AB*, and from the centre *C*, at the distance *CD*, describe (Post. 3) the circle *FDG*, cutting *AB* in *F* and *G*. Bisect (I. 10) *FG* in *H*, and join *CH*. *CH* is perpendicular to *AB*.

Because in the triangles *FHC*, *GHC*, *CF* is equal (Def. 30) to *CG*, *FH* to *GH*, and *CH* common, therefore (I. 4) the angles *CHF*, *CHG*, opposite the equal sides, are equal. Hence (Def. 8) *CH* is perpendicular to *AB*.

ANALYSIS.

$$\begin{tikzcd}
\text{(By const.),} & FH=GH \arrow[dr] & \\
\text{(Def. 30),} & CF=CG \arrow[r] & \text{(I. 4), } CHF=CHG. \\
& CH \text{ common.} \arrow[ur] &
\end{tikzcd}$$

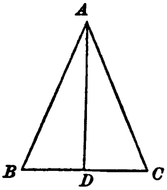
Proposition 13.

***Theorem.**—In an isosceles triangle the angles opposite the equal sides are equal to each other.*

Let ABC be an isosceles triangle having AB equal to AC; then will the angle ABC be equal to the angle ACB.

Bisect (I. 9) the angle BAC by AD, cutting BC in D.

Then because in the triangles ADB, ADC, AB is equal (Hyp.) to AC, the angle BAD to the angle CAD, and AD common, the angles ABD, ACD (I. 7) are equal; and these are the angles of the isosceles triangle opposite the equal sides.



ANALYSIS.

$$\begin{tikzcd}
\text{(By Hyp.),} & AB=AC \arrow[dr] & \\
\text{(By constr.),} & BAD=CAD \arrow[r] & \text{(I. 7), } ABC=ACB. \\
& AD \text{ common.} \arrow[ur] &
\end{tikzcd}$$

***Corollary.**—Every equilateral triangle is also equiangular.*

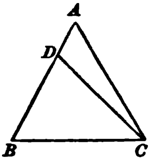
Proposition 14.

***Theorem.**—If two angles of a triangle be equal to each other, the sides which are opposite to them will also be equal to each other.*

Let ABC be a triangle having the angle ABC equal to the angle ACB; then will AB be equal to AC.

For if AB be not equal to AC, let one of them, as AB, be the greater. Cut off (I. 1) DB equal AC.

Then because in the triangles DBC, ACB, DB is equal AC, BC common, and the angle DBC equal (Hyp.) to the angle ACB, therefore (I. 7) the triangle DBC is equal to the triangle ACB, the less to the greater, which is impossible (Ax. 9). Therefore, AB is not unequal to AC; that is, it is equal to it.



ANALYSIS.

$$\left.\begin{array}{ll}\text{(By supposition)} & DB = AC \\ \text{(By Hyp.)} & DBC = ACB \\ & BC \text{ common}\end{array}\right\} \longrightarrow \text{(I. 7) } \triangle DBC = \triangle ACB.$$

***Corollary.**—Every equiangular triangle is also equilateral.*

Proposition 15.

***Theorem.**—The angles which one straight line makes with another on the same side of it, are together equal to two right angles.*

Let AB make with CD the two adjacent angles ABD, ABC, and let R denote a right angle; then will $ABD + ABC = 2R$.

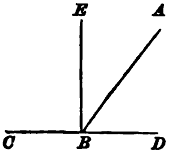
Draw (I. 11) BE at right angles to CD.

Then $CBE = R$ and $EBD = R$. Also

$$ABC = CBE + ABE = R + ABE,$$

and $$ABD = EBD - ABE = R - ABE.$$

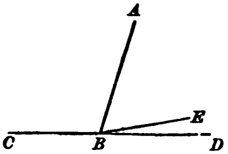
$\therefore$ (Ax. 2) $ABC + ABD = 2R$.



Proposition 16.

***Theorem.**—If at a point in a straight line, two other straight lines on opposite sides of it, make the adjacent angles equal to two right angles, these two lines are in the same straight line.*

At the point B in the straight line AB let BC, BD be drawn, making the sum of the angles ABC, ABD equal to two right angles; then will BC and BD be in the same straight line.



For if BD be not in the same line with CB, suppose BE to be in the same line with CB.

Then (I. 15) $ABC+ABE=2R$;
but (Hyp.) $ABC+ABD=2R$.
$\therefore$ (Ax. 1) $ABC+ABE=ABC+ABD$.

Taking from both ABC, then $ABE=ABD$, the less to the greater, which is impossible. Therefore, BE is not straight with CB, and no line but BD is. Therefore, CBD is a straight line.

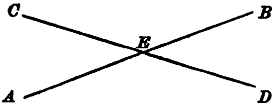
ANALYSIS.

$$\begin{tikzcd}
\left.\begin{array}{l}\text{(By supposition) } ABC\\ \quad +ABE=2R\end{array}\right\} \arrow[dr] & & \\
& \left\{\begin{array}{l} ABC+ABE\\ =ABC+ABD\end{array}\right\} \arrow[r] & \text{(Ax. 3) } ABE=ABD. \\
\left.\begin{array}{l}\text{(By Hyp.) } ABC+ABD\\ \quad =2R\end{array}\right\} \arrow[ur] & ABC=ABC \arrow[ur] &
\end{tikzcd}$$

Proposition 17.

Theorem.—If two straight lines cut each other, the vertical or opposite angles are equal.

Let AB and CD cut each other at E; then will AEC be equal to BED, and CEB to AED.



Because CEA, CEB are adjacent angles, they are together equal to two right angles (I. 15); for the same reason CEB, BED are together equal to two right angles; therefore (Ax. 1), CEA and CEB are equal to CEB and BED. Take away from both the common angle CEB, and the remainders AEC, BED (Ax. 3) are equal.

In the same manner it may be proved that CEB is equal to AED.

ANALYSIS.

$$\left.\begin{array}{l}(\text{I. }15)\ AEC+CEB=2R\\ (\text{I. }15)\ DEB+CEB=2R\end{array}\right> \left.\begin{array}{c}AEC+CEB=DEB+CEB\\ CEB=CEB\end{array}\right> \left\{\begin{array}{c}(\text{Ax. }3)\\ AEC=DEB.\end{array}\right.$$

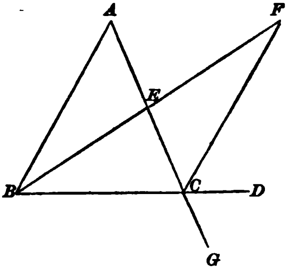
Corollary 1.—Hence, all the angles at the intersection of two straight lines, are equal to four right angles.

Corollary 2.—Also, all the angles made by any number of lines meeting in a point, are equal to four right angles.

Proposition 18.

***Theorem.**—If one side of a triangle be produced, the exterior angle is greater than either of the opposite interior angles.*

Let the side *BC* of the triangle *ABC* be produced to *D*; then will *ACD* be greater than either *ABC* or *BAC*.



Bisect (I. 10) *AC* in *E*; join *BE*, and produce it, making *EF* equal to *BE*; join *FC*.

Because in the triangles *AEB*, *CEF*, *AE* is equal to *EC*, *BE* to *EF*, and the angle *AEB* to the angle *CEF* (I. 17), therefore (I. 7) the angle *BAE* is equal to the angle *ECF*. But *ECF* is less than *ECD*. Therefore *ECD* or *ACD* is greater than *BAC*.

By producing *AC* to *G*, we may prove similarly that *BCG* (or *ACD*) is greater than *ABC*.

ANALYSIS.

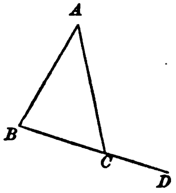
$$\left.\begin{array}{ll}(\text{By Const.}), & AE=EC\\ \text{"} & BE=EF\\ (\text{I. }17), & AEB=FEC\end{array}\right> \left.\begin{array}{l}(\text{I. }7),\ BAE=ECF\\ \qquad\ \ ACD>ECF\end{array}\right> ACD>BAC.$$

Proposition 19.

***Theorem.**—Any two angles of a triangle are together less than two right angles.*

Let ABC be a triangle; any two of its angles are together less than two right angles.

Produce BC to D.



Then, because ACD is the exterior angle of the triangle ABC, it is greater than ABC. To each add ACB; then

$$ACD+ACB>ABC+ACB;$$

but (I. 15), $ACD+ACB=2R$;

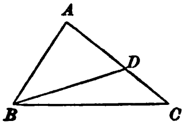
$\therefore$ $ABC+ACB<2R$.

In the same way the theorem may be proved of any two angles of the triangle.

Proposition 20.

***Theorem.**—The greater side of every triangle has the greater angle opposite to it.*

Let ABC be a triangle, in which AC is greater than AB; then will ABC be greater than ACB.



Cut off (I. 1) from AC, AD equal to AB, and join DB.

Because ADB is an exterior angle of the triangle DBC, it is greater (I. 18) than the opposite interior angle DCB. But (I. 13) $ADB=ABD$; $\therefore$ $ABD>ACB$; therefore still more $ABC>ACB$.

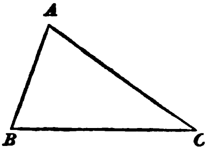
ANALYSIS.

$$\left.\begin{array}{l}\left.\begin{array}{ll}\text{(I. 18),} & ADB>ACB\\ \text{(I. 13),} & ADB=ABD\end{array}\right\rangle ABD>ACB\\ \qquad\qquad\text{(Ax. 9), } ABC>ABD\end{array}\right\rangle ABC>ACB.$$

Proposition 21.

Theorem.—*The greater angle of every triangle has the greater side opposite to it.*

Let ABC be a triangle in which the angle ABC is greater than the angle ACB; then will AC be greater than AB.

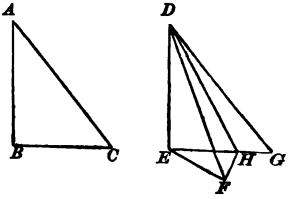


For, if not, it must be equal or less. It is not equal, because then ABC would be equal to (I. 13) ACB. It is not less, for then ABC would be less than (I. 20) ACB. Therefore it is greater.

Proposition 22.

Theorem.—*If two triangles have two sides of one equal to two sides of the other, each to each, but the angles contained by these sides unequal, the third side of that which has the greater angle will be greater than the third side of the other.*

Let ABC, DEF be two triangles, having the sides BA, AC equal to ED, DF, each to each, and the angle BAC greater than EDF; then will BC be greater than EF.



Construct (I. 3) the triangle DEG, having its sides equal to the sides of ABC. Bisect (I. 9) the angle FDG by DH, meeting EG in H; join FH.

Because DG and DF are both equal to AC, they are equal to each other. Then in the triangles DHF, DHG, DF is equal to DG, DH common, and the angle FDH equal to GDH, therefore (I. 7) FH is equal to GH.

Now (I. 2), $EH+HF>EF$, or $EH+HG>EF$;

$\therefore$ EG or BC is greater than EF.

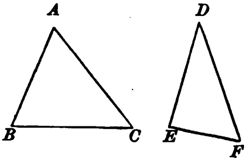
ANALYSIS.

$$\begin{tikzcd}
(\text{Ax. }1),\ DF=DG \arrow[dr] & & & \\
(\text{Constr.}),\ FDH=GDH \arrow[r] & (\text{I. }7),\ FH=GH \arrow[dr] & & \\
DH\text{ common.} \arrow[ur] & & EG>EF \arrow[dr] & \\
& (\text{I. }2),\ EH+HF>EF \arrow[ur] & & BC>EF. \\
& & (\text{Constr.}),\ EG=BC \arrow[ur] &
\end{tikzcd}$$

Proposition 23.

***Theorem.**—If two triangles have two sides of one equal to two sides of the other, each to each, but the third sides unequal, the angle contained by the two sides of the one which has the greater third side, will be greater than the angle contained by the two sides of the other.*

Let *ABC*, *DEF* be two triangles having *AB*, *AC* equal to *DE*, *DF*, each to each, and *BC* greater than *EF*; then will *BAC* be greater than *EDF*.



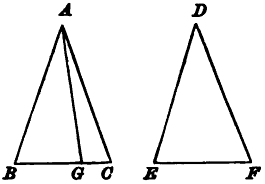
For, if not, it must be either equal to it or less. It is not equal, for then *BC* would be equal to (I. 7) *EF*. It is not less, for then *BC* would be less than (I. 22) *EF*. But *BC* is greater than *EF* (Hyp.), therefore *BAC* is greater than *EDF*.

Proposition 24.

***Theorem.**—If two triangles have two angles and an opposite side of one, equal to two angles and an opposite side of the other, each to each, the equal sides being opposite equal angles, the triangles will be equal in all their parts.*

Let *ABC*, *DEF* be two triangles which have *ABC*, *ACB* and *AB* of one, equal to *DEF*, *DFE* and *DE* of the other, each to each; then will the triangles be equal in all their parts.

For if *BC* be not equal to *EF*, one of them must be greater. Let *BC* be the greater, and cut off (I. 1) *BG* equal to *EF*. Then in the triangles *ABG*, *DEF*, *AB*, *BG* and *ABG* are equal, each to each, to *DE*, *EF* and *DEF*; therefore (I. 7) *AGB* is equal to *DFE*; but *DFE* is equal (Hyp.) to *ACB*; therefore *AGB* is equal to *ACB*, an exterior angle of the triangle *AGC* equal to an opposite interior angle *ACG*, which is impossible (I. 18). Therefore *BC* is not unequal to *EF*; that is, it is equal to it, and the other parts of the triangles are equal by (I. 7).



NOTE.—Let the student write out the complete analysis.

Scholium.—It will be observed, by comparing Props. (4), (7), (8), (24), that there are three cases in which triangles may be proved equal in all their parts:

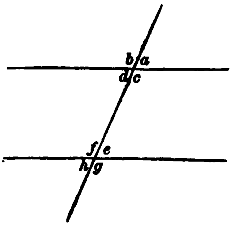
1. When they have three sides of one equal to three sides of the other, each to each (I. 4);

2. When they have two sides and included angle of one equal to two sides and included angle of the other, each to each (I. 7);

3. When they have two angles and a side of one equal to two angles and a side of the other, each to each (I. 8), (I. 24).

DEFINITION.

If a line meet two parallel lines, the various angles have different names.



1. *d* and *e*, taken in relation to each other, are called *alternate*, as also *c* and *f*.

2. *c* and *e*, or *d* and *f*, are *interior angles on the same side.*

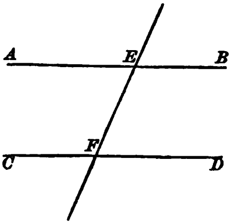
3. *a* and *e*, or *b* and *f*, are respectively *exterior* and *opposite interior angles on the same side.*

Proposition 25.

Theorem.—If a straight line falling on two straight lines, make the alternate angles equal to each other, the two straight lines will be parallel.

Let *EF*, falling on *AB* and *CD*, make the angles *AEF*, *EFD* equal to each other; then will *AB* and *CD* be parallel.

For if they be not parallel, they will meet if produced. Let them meet towards *B* and *D*. Then *E*, *F* and the point of meeting will be the three angles of a triangle of which *AEF* is an exterior angle; therefore *AEF* is greater (I. 18) than *EFD*; but (Hyp.) *AEF* is equal to *EFD*. Hence *AB* and *CD* will not meet toward *B* and *D*. In the same way it may be proved they will not meet toward *A* and *C*. They are therefore parallel (Def. 4).



Proposition 26.

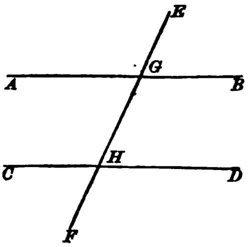
Theorem.—Two straight lines will be parallel—

1. When the exterior and opposite interior angle on the same side are equal to each other;

2. When the two interior angles on the same side are equal to two right angles.

1. Let *EF*, falling on *AB* and *CD*, make *EGB* equal to *GHD*; *AB* and *CD* will be parallel.

For *EGB* is equal (I. 17) to *AGH*; therefore *AGH* is equal to *GHD*; but they are alternate angles; therefore *AB* and *CD* are parallel (I. 25).



2. Let *BGH*, *GHD* be together equal to two right angles; then will *AB* and *CD* be parallel.

BGH and *AGH* are equal (I. 15) to two right angles; therefore

$$BGH + AGH = BGH + GHD.$$

Taking away the common angle *BGH*, then $AGH = GHD$, and they are alternate angles; therefore *AB* and *CD* are parallel (I. 25).

Proposition 27.

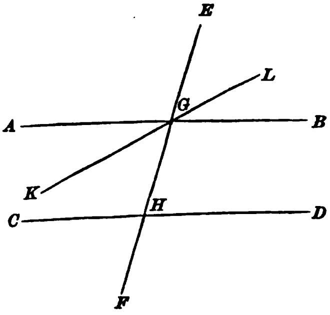
***Theorem.**—If a straight line fall on two parallel straight lines, it makes—*

1. The alternate angles equal to each other.

2. The exterior angle equal to the opposite interior angle on the same side.

3. The two interior angles on the same side together equal to two right angles.

Let the straight line *EF* fall on the two parallel lines *AB*, *CD*.



1. The alternate angles *AGH*, *GHD* will be equal to each other.

For if *AGH* be not equal to *GHD*, draw (I. 5) *KGH* equal to *GHD*.

Then *KL* is parallel (I. 25) to *CD*. But (Hyp.) *AB* is parallel to *CD*. Hence, through the point *G* two lines, *AB* and *KL*, have been drawn, both parallel to *CD*, which is impossible (Ax. 11). Hence *AGH* is not unequal to *GHD*.

2. The exterior angle *EGB* will be equal to the opposite interior angle *GHD*.

For *EGB* is equal (I. 17) to *AGH*, and *AGH* has been proved equal to *GHD*; therefore *EGB* is equal to *GHD*.

3. The interior angles *BGH*, *GHD* are together equal to two right angles.

For *EGB* is equal to *GHD*; add to both *BGH*, and $EGB + BGH = BGH + GHD$; but (I. 15) $EGB + BGH = 2R$;

$$\therefore \quad BGH + GHD = 2R.$$

Corollary 1.—*KL and CD will meet on that side of EF on which the interior angles are less than two right angles.*

Let *KGH*, *GHC* be less than two right angles. If *KL* and *CD* do not meet toward *K* and *C*, they are parallel, or they meet toward *L* and *D*. They are not parallel, for then *KGH* and *GHC* would be equal to two right angles. Neither do they meet toward *L* and *D*; for then *LGH* and *GHD* would be two angles of a triangle, and less than two right angles. But if *KGH*, *GHC* are less than two right angles, *LGH*, *GHD* are greater. Hence *KL* and *CD* will meet toward *K* and *C*.

Corollary 2.—*If a straight line is at right angles to one of two parallel lines, it is also at right angles to the other.*

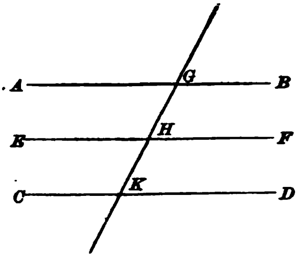
For if *BGH* is a right angle, *GHD* will also be a right angle; since they are together equal to two right angles.

Proposition 28.

Theorem.—*Straight lines which are parallel to the same straight line are parallel to each other.*

Let *AB*, *CD* be each of them parallel to *EF*; *AB* is parallel to *CD*.

Draw *GHK* cutting all three.

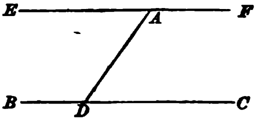


Then, because *AB*, *EF* are parallel, the alternate angles *AGH*, *GHF* are equal (I. 27) to each other; and because *EF*, *CD* are parallel, the exterior angle *GHF* is equal (I. 27) to the opposite interior angle *HKD*. Therefore *AGH*, *HKD* are equal to each other; and they are alternate; hence *AB* and *CD* are parallel (I. 25).

Proposition 29.

***Problem.**—To draw a straight line through a given point parallel to a given straight line.*

Let *A* be the point, and *BC* the line; it is required to draw through *A* a line parallel to *BC*.

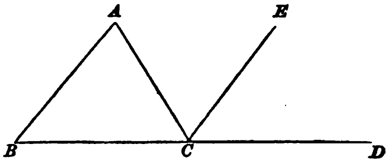


In *BC* take any point *D*, and join *AD*; at *A* make (I. 5) the angle *DAE* equal to the angle *ADC*, and produce *EA* to *F*.

Then, because the alternate angles *EAD*, *ADC* are equal, *EF* is parallel (I. 25) to *BC*.

Proposition 30.

***Theorem.**—If one side of a triangle be produced, the exterior angle is equal to the two opposite interior angles; and the three interior angles of any triangle are equal to two right angles.*



Let *ABC* be a triangle, and let *BC* be produced to *D*; then will *ACD* be equal to *BAC* and *ABC*, and

BAC, *ABC* and *ACB* will be equal to two right angles.

From *C* draw (I. 29) *CE* parallel to *BA*.

Then because *AB* and *CE* are parallel, and *AC* meets them, the alternate angles *BAC*, *ACE* are equal to each other; and because *BCD* meets them, the exterior angle *ECD* is equal to the opposite interior angle *ABC*; therefore the whole angle *ACD* is equal to the two, *BAC*, *ABC*.

To each of these equals add *ACB*;

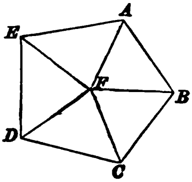
then $ACD + ACB = BAC + ABC + ACB.$

But $ACD + ACB = 2R;$

$\therefore$ $BAC + ABC + ACB = 2R.$

***Corollary 1.**—All the interior angles of a polygon are together equal to twice as many right angles as the figure has sides, minus four right angles.*

Let *ABCDE* be a polygon, and let *n* represent the number of its sides. Draw lines from the angles to any point, *F*, within the polygon. These will form *n* triangles. All the angles of these triangles will be equal to 2*n* right angles, since the angles of each are equal to 2 right angles. Now the angles at *F* are together equal (I. 17, Cor. 2) to 4 right angles. Therefore the other angles at *A*, *B*, etc. are equal to $2n - 4$ right angles.



Scholium to Corollary 1.—If the figure be a triangle, *n* would be 3, and the angles would be

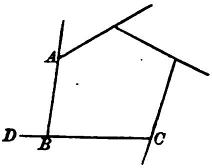
$$2n - 4 = 2 \times 3 - 4 = 2R;$$

for a quadrilateral, $2 \times 4 - 4 = 4R;$

for a pentagon, $2 \times 5 - 4 = 6R,$ and so on.

***Corollary 2.**—If the sides of a polygon be produced in one direction only, all the exterior angles are together equal to four right angles.*

For an exterior and interior angle, as *ABD* and *ABC*, are together equal (I. 15) to two right angles. Hence all the exterior and interior angles are together equal to $2n$ right angles. But the interior (Cor. 1) are equal to $2n-4$ right angles; hence the exterior are equal to 4 right angles.



***Corollary 3.**—If two angles of a triangle be given, or only their sum, the third may be found by subtracting their sum from two right angles.*

***Corollary 4.**—If two angles of one triangle be equal to two angles of another, the third angles will be equal.*

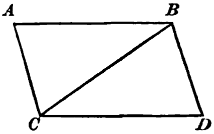
***Corollary 5.**—In any triangle there can be but one right angle, and the other angles will be together equal to a right angle.*

Scholium.—Angles are measured in degrees (°), minutes (′) and seconds (″); a right angle contains 90°.

Proposition 31.

***Theorem.**—If two sides of a quadrilateral be equal and parallel, the other sides will also be equal and parallel, and the figure will be a parallelogram.*

Let *ABDC* be a quadrilateral having *AB* equal and parallel to *CD*; then will *AC* be equal and parallel to *BD*.



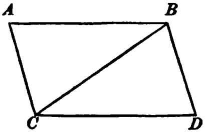
Join *BC*; then because *AB* and *CD* are parallel, the alternate angles *ABC* and *BCD* are equal (I. 27). Hence in the triangles *ABC*, *BCD*, *AB*, *BC* and the angle *ABC* are equal to *DC*, *CB* and the angle *DCB*, each to each; therefore *AC* is equal (I. 7) to *BD*, and the angle *ACB* to the angle *CBD*; and they are alternate angles;

therefore *AC* is parallel (I. 25) to *BD*, and *ABDC* is a parallelogram (Def. 22).

Proposition 32.

***Theorem.**—The opposite sides and angles of a parallelogram are equal to each other, and the diagonal bisects it.*

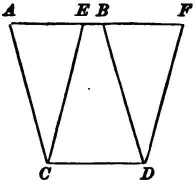
Let *ABDC* be a parallelogram; then will its opposite sides and angles be equal and the diagonal *CB* bisect it.



Because *AB*, *CD* are parallel (Def. 22), the alternate angles *ABC*, *BCD* are equal (I. 27), and because *AC*, *BD* are parallel, the angles *ACB*, *CBD* are equal (I. 27), and *CB* is common to the triangles *ABC*, *CBD*; hence (I. 8), the angle *A* is equal to the angle *D*, the parts of *B* respectively equal to the parts of *C*, *AC* to *BD*, *AB* to *CD*, and the area of *ABC* to the area of *BCD*.

Proposition 33.

***Theorem.**—Parallelograms on the same base and between the same parallels are equal.*



Let *ABDC*, *EFDC* be two parallelograms, on the same base *CD* and between the same parallels *CD*, *AF*; they will be equal to each other

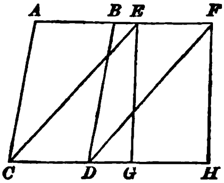
AC and *BD* are equal because they are opposite sides of a parallelogram (I. 32); as also *EC* and *FD*. Also *AB* and *EF* are equal, being both equal to *CD* (Ax. 1); subtract these equals from *AF*; whence the remainders *AE* and *BF* are equal (Ax. 3). Hence, the triangles *AEC*, *BFD* are mutually equilateral; they are therefore equal (I. 4) in area. Subtract each of these equals from the whole trapezoid *ACDF*, and the remainders, *ABDC*, *EFDC*, are equal (Ax. 3).

Proposition 34.

***Theorem.**—Parallelograms on equal bases and between the same parallels are equal.*

Let *ABDC*, *EFHG* be parallelograms on equal bases *DC*, *HG*, and between the same parallels *CH*, *AF*; they will be equal.

Join *CE*, *DF*; then



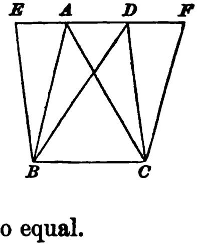
because (Hyp.) $CD = GH$,
and (I. 32.) $GH = EF$,
∴ (Ax. 1) $CD = EF$.

Hence (I. 31), *EFDC* is a parallelogram. Therefore *EFDC*, *ABDC* are equal (I. 33), being on the same base *CD* and between the same parallels *CD*, *AF*. Also *EFDC* is equal to *EFHG*, being on the same base *EF* and between the same parallels *EF*, *CH*. Hence (Ax. 1) *ABDC* is equal to *EFHG*.

Proposition 35.

***Theorem.**—Triangles on the same base and between the same parallels are equal.*

Let *ABC*, *DBC* be two triangles on the same base *BC* and between the same parallels *BC*, *AD*; they will be equal.



Draw (I. 29) *BE* parallel to *CD*, and *CF* parallel to *BA*, meeting *AD* produced in *E* and *F*.

Then *EBCD*, *ABCF* are parallelograms (I. 31), and are equal to each other (I. 33). Hence the triangles *DBC*, *ABC*, halves (I. 32) of these equal parallelograms, are also equal.

***Corollary.**—If a triangle and a parallelogram be on the same base and between the same parallels, the parallelogram will be double the triangle.*

EBCD is evidently double *DBC* or *ABC*.

Proposition 36.

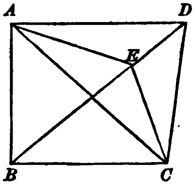
***Theorem.**—Triangles on equal bases and between the same parallels are equal.*

(Construct as in last Proposition parallelograms on the bases of the triangles, of which the triangles will be halves, and apply Prop. 34.)

Proposition 37.

***Theorem.**—Equal triangles on the same base and on the same side of it are between the same parallels.*

Let ABC, DBC be equal triangles on the same base BC; they will be between the same parallels.

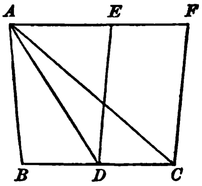


For if AD be not parallel to BC, draw AE parallel to BC and join EC.

Then ABC, EBC are equal (I. 35). But ABC, DBC are equal (Hyp.). Therefore EBC is equal to DBC, the part to the whole; which is impossible (Ax. 9). Hence AE is not parallel to BC, and no line but AD is parallel to BC.

Proposition 38.

***Problem.**—To describe a parallelogram equal to a given triangle.*



Let ABC be a triangle; it is required to describe a parallelogram equal to it.

Bisect BC in D; draw AF parallel to BC, and DE to any point in AF; draw CF parallel to DE; $EDCF$ is equal to ABC.

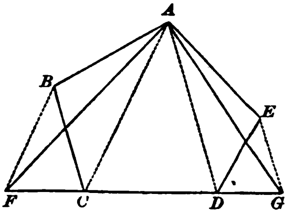
Because ABD, ADC are equal (I. 36), ABC is double ADC; also $EDCF$ is double ADC (I. 35, Cor.). Hence $EDCF$ is equal to ABC.

Scholium.—The angle CDE may be made equal to a right angle or any given angle (I. 5).

Proposition 39.

Problem.—To describe a triangle equal to a given polygon.

Let $ABCDE$ be a polygon; it is required to describe a triangle equal to it.



Join AC, and draw BF parallel to it, and join AF. Then, because ABC, AFC are on the same base, AC, and between the same parallels, AC, BF, they are equal (I. 35).

Similarly, AED, AGD are equal. Add to both sets of these equals the triangle ACD, and $ABCDE$ is equal to AFG.

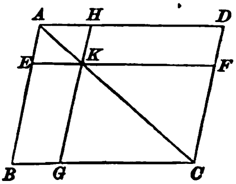
Corollary.—To draw a parallelogram equal to a given polygon.

Construct a triangle equal to the polygon, and then a parallelogram equal to the triangle (I. 38).

Proposition 40.

Theorem.—The complements of a parallelogram are equal to each other.

Let $ABCD$ be a parallelogram, and EH, GF parallelograms about the diagonal; then the complements BK, KD, which make up the whole figure, are equal to each other.



Because AC is a diagonal, the triangles ABC, ADC are equal to each other. For the same reason AEK, AHK, and also KGC, KFC, are equal to each other.

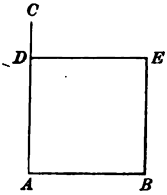
Hence, $\triangle AHK + \triangle KFC = \triangle AEK + \triangle KGC$.

Subtracting these equals from the whole triangles *ABC*, *ADC*, the remainders, *BK*, *KD*, will be equal (Ax. 3).

Proposition 41.

***Problem.**—To describe a square on a given finite straight line.*

Let *AB* be a straight line; it is required to describe a square upon it.



From *A* draw (I. 11) *AC* at right angles to *AB*; and make (I. 1) *AD* equal to *AB*; through *D* draw (I. 29) *DE* parallel to *AB*; and through *B*, *BE* parallel to *AD*; *ADEB* is a square.

For it is a parallelogram (Const.). Hence *AB*, *DE* are equal (I. 32), and *AD*, *BE* are equal; but *AB*, *AD* are equal; hence it is equilateral. Also, because *AB*, *DE* are parallel, the angles *BAD*, *ADE* are together equal (I. 27) to two right angles. But *BAD* is a right angle (Const.) Hence *ADE* is a right angle; and because the figure is a parallelogram, the opposite angles are also right angles (I. 32). Therefore the figure is equilateral and rectangular, and is therefore a square.

***Corollary.**—If one angle of a parallelogram be a right angle, all the angles are right angles.*

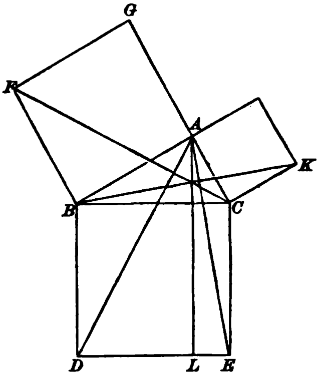
Proposition 42.

***Theorem.**—In any right-angled triangle the square described on the side subtending the right angle, is equal to the sum of the squares described on the sides containing the right angle.*

Let *ABC* be a right-angled triangle, right angled at *A*; then will,—

$$BC^2* = AB^2 + AC^2*.$$

On BC describe (I. 41) the square BE; and on BA, AC the squares BG, AK. Through A draw AL parallel to BD, and join AD, FC. Because BAC, BAG are right angles, CA, AG are in the same straight line (I. 16).



The angle FBA is equal to the angle CBD; to each add ABC; then the angle FBC is equal to the angle ABD. Then, because in the triangles FBC, ABD, FB, BC, and the angle FBC are equal to AB, BD, and the angle ABD, each to each, the triangles are equal (I. 7) in area.

But FBC is half the square BG, because they are on the same base, FB, and between the same parallels, FB, GC. Similarly, ABD is half the parallelogram BL. Therefore the square BG is equal (Ax. 7) to the parallelogram BL.

In the same manner it may be proved that AK is equal to CL. Therefore the sum of BG and AK is equal to the whole square BE, or

$$BC^2 = AB^2 + AC^2.$$

Corollary 1.—Hence $BC^2 - AB^2 = AC^2$.

Corollary 2.—Also, if $AB = AC$,

$$BC^2 = 2AB^2 = 2AC^2.$$

Corollary 3.—*And if two right-angled triangles have two sides of one equal to the corresponding sides of the other, they will be mutually equilateral and equal in all their parts.*

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* This expression means the square described on BC, etc.

EXERCISES ON BOOK I.

The proofs or solutions of the following propositions are to be worked out by the student, by the use of those previously demonstrated. In some cases the lines necessary to the construction are drawn, and the propositions which may be needed are referred to. In others, parts of the solution are given. In all cases they are intended to be so clear that a student may, unaided, complete the proof by the exercise of thought and skill. It will often assist him in obtaining a construction to suppose the figure formed, and to note the properties involved; then, by using these properties, he may form the construction.

1. Upon a given straight line to describe an isosceles triangle having each of the equal sides equal to a given straight line.

2. If the angles at the base of an isosceles triangle be bisected, the portions of the bisecting lines between their intersection and the angles bisected, will be equal (I. 14).

3. If a point be taken without a given straight line, the perpendicular is the shortest line that can be drawn from the point to the line (I. 19), (I. 21).

4. The diagonal of a rhombus bisects its angles (I. 4).

5. The four sides of any quadrilateral are together greater than the two diagonals (I. 2).

6. Construct a triangle having given two angles and a side opposite one of them.

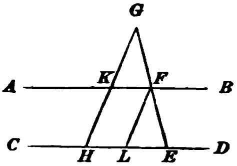
7. Show why we cannot construct a triangle when we have only the three angles:—what ambiguity is there when we have two sides and an angle opposite one of them?

8. A straight line parallel to the base of an isosceles triangle makes equal angles with the sides (I. 27).

9. A line joining two parallels is bisected. Show that any line through the point of bisection and terminating in the parallels is also bisected (I. 27), (I. 8).

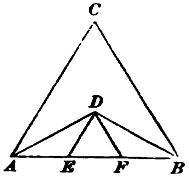
10. If AB and CD be parallel and $FG = EF$, show that any other line GH is bisected at K.

Draw FL parallel to GH, and work in the triangles FLE, GKF.



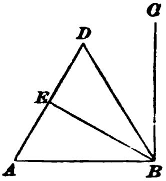
11. Trisect a given straight line.

On the line AB construct an equilateral triangle ABC. Bisect the angles at A and B by AD, BD. Draw DE, DF parallel to AC, CB. Prove that $AE = EF = FB$.



12. Trisect a right angle.

Let ABC be the right angle. On AB construct an equilateral triangle ABD. Bisect the angle ABD by BE. Prove that $ABE = EBD = DBC$ (I. 30).



13. What is the magnitude of an angle of a regular decagon (I. 30 Cor.)?

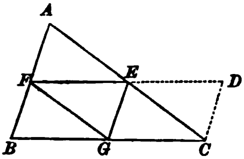
14. What is the magnitude of an exterior angle of a regular hexagon (I. 30 Cor.)?

15. If the opposite sides of a quadrilateral be equal, it is a parallelogram.

16. If the opposite angles of a quadrilateral be equal, it is a parallelogram.

17. The four triangles into which a parallelogram is divided by its diagonals, are equal in area.

18. If the middle points of the three sides of a triangle be joined by straight lines, the four triangles formed will be equal in area.

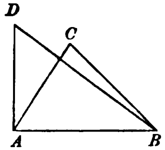


Draw CD parallel to AB and produce FE to D. Show now the equality of the triangles AEF, DEC. Hence EF is parallel to BC. Similarly, FG is parallel to AC, and EG to AB. Hence the proposition may be deduced.

19. Given the middle points of the three sides of a triangle, to construct the triangle.

20. Given the middle points of the four sides of a parallelogram, to construct the parallelogram.

21. The square on a side of a triangle subtending an acute angle, is less than the squares on the other sides.



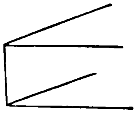
Let BAC be an acute angle, then $BC^2 < AC^2 + AB^2$.

Make $AD = AC$, and at right angles to AB (I. 22).

22. The square on the side subtending the obtuse angle, is greater than the squares on the other sides.

23. Any side of a triangle is greater than the difference between the other two.

24. Two angles are equal if their sides be parallel, each to each, and lying in the same direction (I. 27), (Ax. 3).



25. To find the side of a square equal in area to two given squares (I. 42).

26. If from the ends of one side of a triangle straight lines be drawn to a point within the triangle, these lines shall contain a greater angle than the other two sides of the triangle (I. 18).

27. If each of the equal angles of an isosceles triangle be double the third angle, then the exterior angle formed by producing one of the equal sides beyond the base is three times the third angle.

28. In the last example how many degrees in the various angles of the figure?

29. If the equal sides of an isosceles triangle be produced beyond the base, the angles on the other side of the base will be equal.

30. If one angle of a parallelogram contain 40°, what is the value of each of the others?

31. If a line be drawn bisecting an angle, any point of it is equally distant from the sides of the angle.

32. The lines bisecting the three angles of a triangle all intersect in the same point.

33. One of the angles of a parallelogram is three halves of a right angle. What are the values of the others in parts of a right angle? in degrees?

34. One of the exterior angles of an equilateral figure is $\frac{2}{3}$ of a right angle. How many sides has the figure?

35. Construct a five-sided figure, four of whose sides are 3, 4, 5, 6, and whose angles in the same order are 2, $\frac{1}{2}$, 1, $\frac{3}{2}$, 1, right angles.

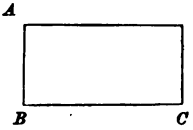
BOOK II.

RECTANGLES.

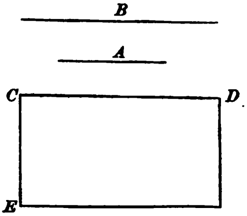
DEFINITIONS.

1. Every right-angled parallelogram, or ***rectangle***, is said to be contained by any two of the straight lines which contain one of the right angles.

Thus, the rectangle AC is said to be contained by AB and BC, and is called the rectangle $AB.BC$, or simply $AB.BC$.

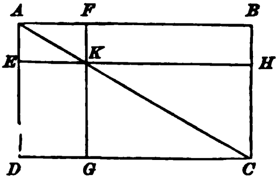


When we speak of the rectangle of two disconnected straight lines, as A and B, we mean the right-angled parallelogram of which those lines are the adjacent sides. Thus, make EC equal to A, and CD, at right angles to it, equal to B, and complete the parallelogram. The rectangle $A.B$ is ED.



By rectangle we mean the figure formed, and not the product of the lines. We will show *hereafter* that the area of the rectangle is equal to the product of the number of units in the two sides.

2. In every parallelogram the figure formed by either of the parallelograms about the diameter, together with the two complements, is called a *gnomon*.

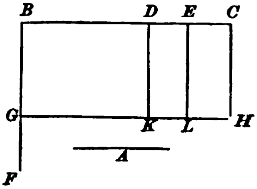


Thus, the parallelogram *EF*, together with the complements *FH*, *DK*, is the gnomon, which is more briefly expressed by the letters *DFH* or *BEG*.

Proposition 1.

***Theorem.**—If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines, is equal to the rectangles contained by the undivided line and the several parts of the divided line.*

Let *A* and *BC* be two straight lines; and let *BC* be divided into any number of parts at the points *D*, *E*; the rectangle contained by the straight lines *A*, *BC*, will be equal to the rectangle contained by *A*, *BD*, together with that contained by *A*, *DE*, and that contained by *A*, *EC*.



From the point *B* draw (I. 11) *BF* at right angles to *BC*; and make (I. 1) *BG* equal to *A*; through *G* draw (I. 29) *GH* parallel to *BC*; and through *D*, *E* and *C* draw (I. 29) *DK*, *EL* and *CH* parallel to *BG*.

Then the rectangle *BH* is equal to the rectangles *BK*, *DL*, *EH*.

But $BH = BG : BC = A . BC$, because *A* is equal to *BG*.

Also, $BK = BG . BD = A . BD$;

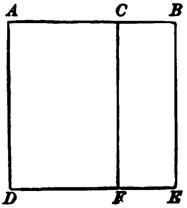
$DL = DK . DE = A . DE$;

$EH = EL . EC = A . EC$;

$\therefore \quad A . BC = A . BD + A . DE + A . EC.$

Proposition 2.

Theorem.—*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the square of the whole line.*



Let the straight line AB be divided into any two parts in the point C; then

$$AB.AC+AB.CB=AB^2.$$

On AB describe (I. 41) the square AE, and draw (I. 29) through C, CF parallel to AD.

Then $AF+CE=AE$;

but $AF=AD.AC=AB.AC$,

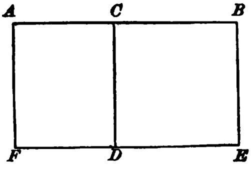
and $CE=CF.CB=AB.CB$,

and $AE=AB^2$;

$\therefore$ $AB.AC+AB.CB=AB^2$.

Proposition 3.

Theorem.—*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the square of the aforesaid part.*



Let the straight line AB be divided in two parts in the point C, then $AB.BC=AC.CB+BC^2$.

Upon BC describe (I. 41) the square CE; produce ED to F, and draw AF parallel to CD.

Then $AE = AD + CE$;

but $AE = AB.BE = AB.BC$;

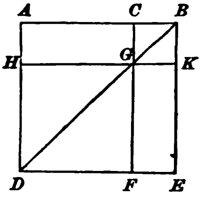
also, $AD = AC.CD = AC.CB$,

and $CE = CB^2$.

$\therefore$ $AB.BC = AC.CB + CB^2$.

Proposition 4.

***Theorem.**—If a straight line be divided into any two parts, the square of the whole line is equal to the squares of the two parts, together with twice the rectangle contained by the two parts.*



Let the straight line AB be divided into any two parts in C; then will

$$AB^2 = AC^2 + BC^2 + 2AC.CB.$$

Upon AB describe (I. 41) the square $ABED$; join BD; through C draw (I. 29) CF parallel to AD, and through G draw (I. 29) HGK parallel to AB.

Because CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal (I. 27) to the interior and opposite angle ADB; but ADB is equal (I. 13) to ABD. Therefore CGB is equal to CBG, and therefore the side CG is equal to (I. 14) the side CB; but CB is equal (I. 32) to GK; therefore the figure $CGKB$ is equilateral. It is also rectangular, for the angle CBK being a right angle, the other angles are also right angles (I. 41, Cor.). Wherefore it is a square upon the side CB. For the same reason HF is a square upon the side HG, which is equal to AC.

The complement AG is equal (I. 40) to the complement GE.

Also, $AG = AC.CG = AC.CB$;

$\therefore$ $AG + GE = 2AC.CB$.

Also, $HF = AC^2$, and $CK = CB^2$;

$\therefore$ $HF + CK + AG + GE = AC^2 + CB^2 + 2AC.CB.$

But $HF + CK + AG + GE = AE = AB^2$;

$\therefore$ $AB^2 = AC^2 + BC^2 + 2AC.CB.$

***Corollary 1.**—Hence it is manifest that the parallelograms about the diagonal of a square are also squares.*

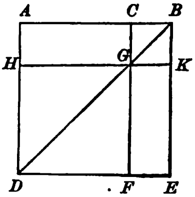
***Corollary 2.**—The square on a line is four times the square on half the line.*

For let C be the middle point of AB, then $AB^2 = AC^2 + CB^2 + 2AC.CB = 4AC^2$.

Proposition 5.

***Theorem.**—The square on the difference of two lines, is equal to the sum of their squares diminished by twice the rectangle of the lines.*

Let AB, BC be two lines: place BC on BA from B toward A; then AC is their difference; then $AC^2 = AB^2 + BC^2 - 2AB.BC.$



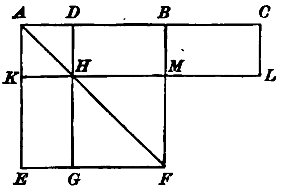
On AB describe the square $ABED$; join DB, and through C draw CF parallel to BE; through G draw HK parallel to AB; then $AE = AB^2$, $CK = CB^2$ and $HF = AC^2$.

Because (I. 40) $AG = GE$, add to both CK; then $AK = CE$, and $AK + CE = 2AK = 2AB.BK = 2AB.BC.$ Now, if from $AE + CK$, $AK + CE$ be taken away, there will remain the square HF; that is, $HF = AE + CK - (AK + CE)$, or $AC^2 = AB^2 + BC^2 - 2AB.BC.$

Proposition 6.

***Theorem.**—The rectangle contained by the sum and difference of two lines is equal to the difference of their squares.*

Let AB, BC be two lines; then $(AB+BC)(AB-BC) = AB^2 - BC^2$.



From BA cut off (I. 1) BD equal to BC, then $AC = AB+BC$ and $AD = AB - BC$. On AB describe the square $AEFB$; join AF, and draw DG parallel to AE; through H draw KHL parallel to AC, and through C draw CL parallel to AK.

Then $EB = AB^2$ and $GM = DB^2$. Now, because $DB = BC$ (I. 34), $DM = BL$; but (I. 40) $DM = KG$, therefore $BL = KG$; add to both AM, and the gnomon $BKG = AL$. Now the gnomon BKG is the difference between the squares EB and GM, or $AB^2 - DB^2$; and the rectangle $AL = AC . AK = AC . AD = (AB+BC)(AB-BC)$; therefore $(AB+BC)(AB-BC) = AB^2 - DB^2$.

***Corollary.**—If a line be bisected and produced, the rectangle of the whole line thus produced and the part produced, together with the square of the line between the points of section, is equal to the square of the line made up of the half and the part produced.*

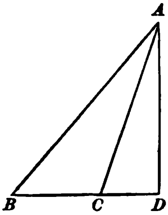
For, CD being bisected in B and produced in A, we have, adding DB^2 to the last equals,

$$AC . AD + DB^2 = AB^2.$$

Proposition 7.

***Theorem.**—In obtuse-angled triangles the square of the side subtending the obtuse angle, is greater than the sum of the squares of the other two sides, by twice the rectangle contained by one of* *the sides adjacent the obtuse angle, and the line, between the obtuse angle and the foot of the perpendicular falling on this side produced, from the opposite angle.*

Let ABC be an obtuse-angled triangle, the obtuse angle ACB, and from A let AD be drawn (I. 12) perpendicular to BC produced; the square of AB is greater than the squares of AC, CB by twice the rectangle $BC.CD$.



Because the straight line BD is divided in two in the point C,

(II. 4), $BD^2 = BC^2 + CD^2 + 2BC.CD$;

add AD^2 to both.

Then $(BD^2 + AD^2) = BC^2 + (CD^2 + AD^2) + 2BC.CD$;

But (I. 42) $AB^2 = BD^2 + AD^2$, and $AC^2 = CD^2 + AD^2$;

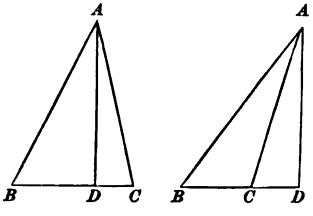
$\therefore$ $AB^2 = BC^2 + AC^2 + 2BC.CD$;

that is, AB^2 is greater than $AC^2 + CB^2$ by $2BC.CD$.

Proposition 8.

Theorem.—*In every triangle the square of the side opposite an acute angle, is less than the sum of the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the line between the acute angle and the foot of the perpendicular falling on this side from the opposite angle.*

Let ABC be a triangle, and the angle B one of its acute angles, and upon BC, one of the sides containing it, let fall (I. 12) the perpendicular AD from the opposite angle; then the square of AC is less than the squares CB, BA by twice the rectangle $CB.BD$.



Because CD is the difference between BC and BD,

(II. 5), $CD^2 = BC^2 + BD^2 - 2BC . BD$;

add AD^2 to both,

$$(CD^2 + AD^2) = BC^2 + (BD^2 + AD^2) - 2BC.BD.$$

But (I. 42) $CD^2 + AD^2 = AC^2$, and $BD^2 + AD^2 = AB^2$;

$\therefore$ $AC^2 = BC^2 + AB^2 - 2BC . BD.$

***Corollary.**—If the square described on one side of a triangle be equal to the sum of the squares of the other two sides, the angle contained by these sides is a right angle.*

For if it were obtuse, the square of the opposite side would be greater, and if acute less, than the sum of the squares of the other sides.

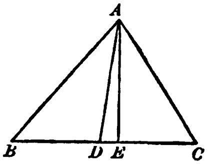
Proposition 9.

***Theorem.**—If one side of a triangle be bisected, the sum of the squares of the other two sides is double of the square of half the line bisected, and of the square of the line drawn from the point of bisection to the opposite angle.*

Let ABC be a triangle, of which the side BC is bisected in D, and DA drawn to the opposite angle.

Then $BA^2 + AC^2 = 2BD^2 + 2DA^2$.

From A draw (I. 12) AE perpendicular to BC.



Now (II. 7), $AB^2 = BD^2 + AD^2 + 2BD . DE$;

and (II. 8), $AC^2 = CD^2 + AD^2 - 2CD . DE.$

Adding these together, and remembering that

$$BD = CD,$$

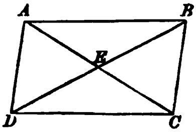
we obtain $AB^2 + AC^2 = 2BD^2 + 2AD^2.$

Proposition 10.

***Theorem.**—The sum of the squares of the diagonals of any parallelogram, is equal to the sum of the squares of the sides of the parallelogram.*

Let $ABCD$ be a parallelogram, and AC, BD its diagonals;

then $$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2.$$



Let AC and BD intersect each other in E; and because the vertical angles AED, CEB are equal (I. 17), and also the alternate angles EAD, ECB (I. 27), the triangles AED, CEB have two angles of each equal, and the sides AD, BC are equal (I. 32); therefore the other sides, which are opposite the equal angles are also equal—namely, AE to EC, and DE to EB.

Since, therefore, DB is bisected in E from (II. 9),

$$AD^2+AB^2=2DE^2+2AE^2.$$

For the same reason,

$$DC^2+CB^2=2DE^2+2EC^2=2DE^2+2AE^2;$$

$$\therefore \quad AD^2+AB^2+DC^2+CB^2=4DE^2+4AE^2.$$

But $4DE^2=DB^2$ (II. 4, Cor. 2), and $4AE^2=AC^2$;

$$\therefore \quad AD^2+AB^2+DC^2+CB^2=DB^2+AC^2.$$

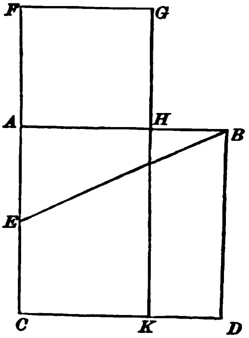
Corollary.—From this demonstration it is manifest that *the diagonals of a parallelogram bisect each other.*

Proposition 11.

***Problem.**—To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts, may be equal to the square of the other part.*

Let AB be a given straight line. It is required to find the point H in it, so that

$$AB\,.\,BH = AH^2.$$



Upon AB describe (I. 41) the square AD; bisect (I. 10) AC in E, and join BE; produce CA to F, making (I. 1) EF equal to EB, and upon AF describe the square FH. AB is then divided in H, so that $AB\,.\,BH = AH^2$.

Produce GH to K. Because AC is bisected in E and produced to F, (II. 6, Cor.),

$$CF\,.\,FA + AE^2 = EF^2 = EB^2.$$

Now (I. 42), $EB^2 = AE^2 + AB^2$; $\therefore\ CF\,.\,FA + AE^2 = AE^2 + EB^2$.

Taking away the common AE^2, and

$$CF\,.\,FA = AB^2.$$

Now, $CF\,.\,FA$ is the rectangle FK,

and AB^2 is the square AD.

Therefore the rectangle FK is equal to the square AD. Taking away the common rectangle AK, the remainder FH is equal to the remainder HD.

But $HD = DB\,.\,BH = AB\,.\,BH$, and $FH = AH^2$;

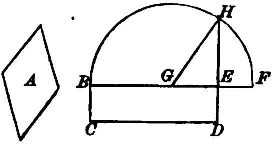
$\therefore\ AB\,.\,BH = AH^2$, and AB is divided in H, so that the rectangle contained by the whole and one part is equal to the square of the other.

Proposition 12.

***Problem.**—To describe a square that shall be equal to a given polygon.*

Let A be a given polygon; it is required to describe a square that shall be equal to it.

Describe (I. 39, Sch.), the rectangle $BCDE$ equal to A. If the sides BE, ED be equal to each other, the thing required is done; but if not, produce one of them, BE to F, and make EF equal to ED, and bisect (I. 10) BF in G; on BF describe the semicircle BHF; produce DE to meet it in H, and join GH.



Because BE is the sum and EF the difference of FG and GE, we have

$$\text{(II. 6)} \qquad BE.EF = FG^2 - EG^2 = GH^2 - EG^2.$$

$$\text{But (I. 42 Cor.)} \qquad GH^2 - EG^2 = EH^2;$$

$$\therefore \qquad BE.EF = EH^2.$$

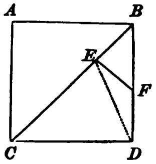
But BD is the rectangle $BE.ED$ or $BE.EF$, and BD is equal by construction to A. Therefore the square on EH is equal to the rectilineal figure A.

EXERCISES.

1. The diagonals of a rhombus bisect each other at right angles.

2. Two parallelograms are equal when they have two sides and the included angle, each to each.

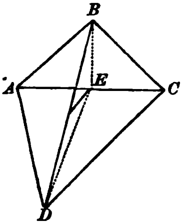
3. The sum of the diagonals of a trapezium is less than the sum of the four lines drawn from any point within the figure to the four angles.



4. If AD be a square, and CE be cut off its diagonal equal to AB, and EF be drawn at right angles to CB, then $BE = EF = FD$.

5. The squares of the diagonals of a tra-

pezium are together less than the squares of the four sides, by four times the square of the line joining the middle points the diagonals.



From (II. 9)

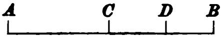
$$AB^2+BC^2=2AE^2+2EB^2,$$

$$AD^2+DC^2=2AE^2+2ED^2.$$

Add and apply (II. 4, Cor. 2) and (II. 9).

6. If a straight line be divided into two equal, and also into two unequal, parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half the line.

If AB be divided equally at C and unequally at D, then



$$AD\,.\,DB+CD^2=CB^2.$$

Describe a square on CB, and proceed as in (II. 6).

7. Divide a given straight line into two parts so that the rectangle contained by them shall be the greatest possible.

Use the property of last exercise, and show that the middle point is the point of division.

8. Construct a rectangle equal to the difference between two given squares.

From Ex. 6 we have $AD\,.\,DB=CB^2-CD^2$. From this the construction may be deduced.

9. The centre of a circle is on the middle point of the base of a triangle; if the vertex be any point of the circumference, the sum of the squares of the two sides is equal to the square of the diameter (II. 9).

10. The square on the perpendicular, drawn from the right angle of a right-angled triangle to the base, is equal to the rectangle contained by the segments of the base.

Bisect the base, and use Exs. 6 and 9.

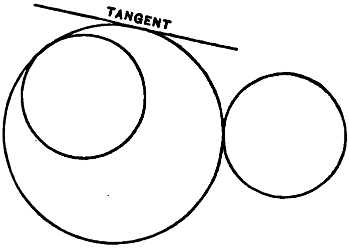
BOOK III.

CIRCLES.

DEFINITIONS.

1. A STRAIGHT line is said to *touch* a circle when it meets the circle, and being produced does not cut it.

And that line which has but one point in common with the circle is called the **tangent,** and the point in common, the **point of contact.**



2. Circles are said to *touch* each other when they meet, but do not cut.

3. Straight lines are said to be *equally distant* from the centre of a circle, when the perpendiculars drawn to them from the centre are equal;

4. And the straight line on which the greater perpendicular falls, is said to be farther from the centre.

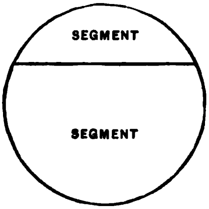
5. An **arc** is any portion of the circumference.

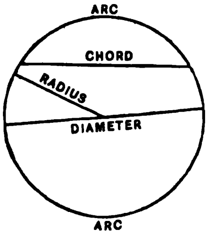
6. The **chord** of an arc is the straight line which joins its extremities.

7. The **segment** of a circle is the portion cut off by a chord.

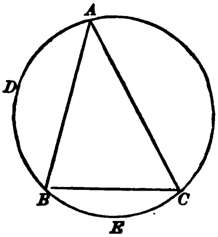
8. A **secant** is a straight line cutting a circle.

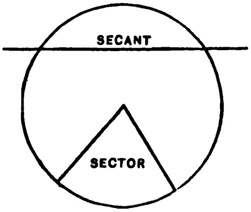
9. A **sector** is the figure contained by two radii and the arc between them.





10. The **angle in a segment,** is the angle contained by two straight lines drawn from any point of the circumference to the extremities of the line which forms the base of the segment.





BAC is an angle in the segment *BDAC*, and standing on the arc *BEC*.

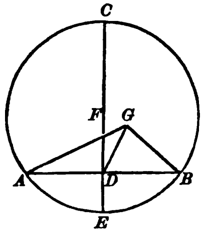
11. An **inscribed figure** is one which has all its angles in the circumference of the circle; and the circle is said to *circumscribe* the figure.

12. An **inscribed circle** is one which touches all the sides of the figure; and the figure is said to *circumscribe* the circle, or be described about it.

Proposition 1.

Problem.—To find the centre of a given circle.

Let *ABC* be the given circle; it is required to find its centre. Draw within it any straight line *AB*, and bisect (I. 10) it in *D*; from the point *D*, draw (I. 11) *DC* at right angles to *AB*, and produce it to *E*, and bisect *CE* in *F*. The point *F* is the centre of the circle.



For if not, let, if possible, *G* be the centre, and join *GA*, *GD*, *GB*. Then, because *DA* is equal to *DB*, and *AG* to *GB*, being radii, and *DG* common to the two triangles *ADG*, *BDG*, therefore the angle *ADG* is equal (I. 4) to the angle *GDB*. Therefore (I. Def. 8) the angle *GDB* is a right angle. But *FDB* is likewise a right angle; wherefore the angle *FDB* is equal to the angle *GDB*, the greater to the less, which is impossible. Therefore, *G* is not the centre of the circle *ABC*. In the same manner it can be shown that no other point but *F* is the centre; that is, *F* is the centre of the circle *ABC*.

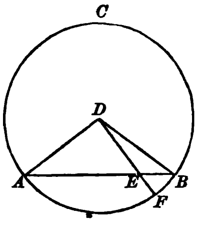
Corollary.—If in a circle a straight line bisect another at right angles, the centre of the circle is in the line which bisects the other.

Proposition 2.

Theorem.—If any two points be taken in the circumference of a circle, the straight line which joins them will fall within the circle.

Let *ABC* be the circle, and *A*, *B* any two points in the circumference; the straight line drawn from *A* to *B* will fall within the circle.

Take *E*, any point in *AB*; join *DE*, and let it meet the circle in *F*. Now, because *DA* is equal to *DB*, *DAB* is equal (I. 13) to *DBA*; and because *DEB* is exterior, it is greater (I. 18) than *DAE*, the opposite interior angle; hence *DEB* is greater than *DBE*, and therefore *DB* is greater than (I. 21) *DE*. Hence *DF* is greater than *DE*, and *E* is within the circle; and the same may be proved of any point of *AB*.



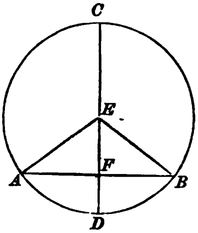
Proposition 3.

Theorem.—*If a straight line through the centre of a circle bisect a line not passing through the centre, it will cut it at right angles; and if it cut it at right angles, it will bisect it.*

Let *ABC* be a circle, and let *CD*, a diameter, bisect *AB*, a chord, in the point *F*; it cuts it at right angles.

Take (III. 1) *E*, the centre of the circle, and join *EA*, *EB*. Then, because in the triangles *AEF*, *BEF*, *AE* = *EB*, being radii, *AF* = *FB* (Hyp.), and *EF* common, therefore (I. 4) *AFE* = *BFE*; and (I. Def. 8) they are right angles.

Again, let *CD* cut *AB* at right angles; then will *AF* = *FB*.



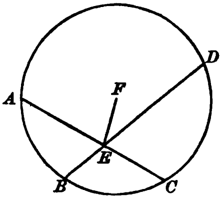
Because *AE* = *BE*, then (I. 13) *EAB* = *EBA*; also the angles at *F* are right angles, and *EF* is common to both triangles; therefore (I. 24) *AF* = *FB*.

Corollary.—*The perpendicular through the middle of a chord, and terminated both ways by the circumference, is a diameter, and its middle point is the centre.*

Proposition 4.

Theorem.—If in a circle two straight lines, which do not pass through the centre, cut each other, they do not bisect each other.

Let $ABCD$ be a circle, and AC, BD two straight lines in it which do not pass through the centre; they will not bisect each other.



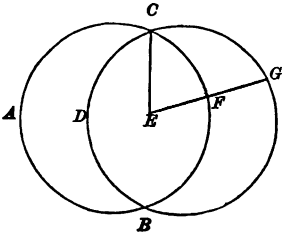
For, if possible, let $AE = EC$ and $BE = ED$.

Take (III. 1) F, the centre of the circle, and join EF; and because EF through the centre bisects AC, the angle FEC is (III. 3) a right angle. For the same reason FED is a right angle. Hence FED is equal to FEC, the less to the greater, which is impossible; therefore AC, BD do not bisect each other.

Proposition 5.

Theorem.—If two circles cut each other, they cannot have the same centre.

Let the two circles ABC, CDG cut each other in the points B, C; they cannot have the same centre.

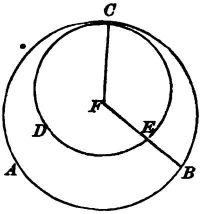


For, if possible, let E be their centre. Join EC, and draw any straight line EFG, meeting the circles. Because E is the centre of ABC, EC is equal to EF; also because E is the centre of CDG, EC is equal to EG; therefore EF is equal to EG, which is impossible. Hence E is not the centre of the circles.

Proposition 6.

Theorem.—If two circles touch each other internally, they cannot have the same centre.

Let ABC, CDE touch each other internally at C; they cannot have the same centre.



For, if possible, let it be F. Join FC, and draw any straight line FEB. Because F is the centre of the circles CDE, ABC, FC is equal to both FE and FB; therefore FE and FB are equal to each other, which is impossible. Hence F is not the centre of both circles.

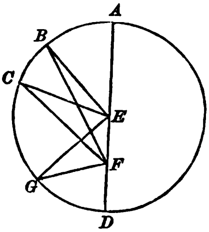
Proposition 7.

Theorem.—If a point be taken in the diameter of a circle which is not the centre, and straight lines be drawn from it to the circumference,—

1. *The greatest is that in which the centre is, and the other part of the diameter is the least.*

2. *Of the others, that which is nearer to the line through the centre, is greater than that more remote.*

Let $ABCD$ be a circle, AD its diameter, F a point in it, and E the centre. FA is the greatest line from F to the circumference, and FD the least; FB is greater than FC, and FC than FG.



1. Join BE, CE, GE.

Then (I. 2) $BE+EF>BF$;
but $BE=AE$, hence $AF>BF$,
also $GF+FE>GE$;
but $GE=DE$;
hence $GF+FE>DE$;
take away EF, and $FG>FD$.

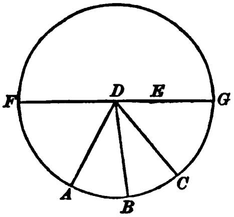
In the same way AF may be proved greater, and FD less, than any other lines from F.

2. Because *BE*, *EF* are equal to *CE*, *EF*, and the angle *BEF* greater than *CEF*, therefore (I. 22) *BF* is greater than *CF*. In the same way *CF* is greater than *GF*.

Proposition 8.

***Theorem.**—If a point be taken within a circle, from which there fall more than two equal straight lines on the circumference, that point is the centre of the circle.*

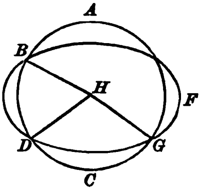
Let *D* be a point in *ABC*, from which three equal lines, *DA*, *DB*, *DC*, are drawn to the circumference; *D* is the centre of the circle.



For, if not, let *E* be the centre; join *DE*, and let it meet the circumference in *F* and *G*. Then because from *D*, a point in the diameter, lines are drawn to the circumference, the greatest is (III. 7) *DG* passing through the centre, and *DC* is greater than *DB*, and *DB* than *DA*. But they are equal (Hyp.); which is impossible. Therefore *E* is not the centre, and no other point but *D* can be the centre.

Proposition 9.

***Theorem.**—One circle cannot cut another in more than two points.*



If possible, let *ABC* cut *BDF* in more than two points, as *B*, *D*, *G*. Take *H*, the centre of the circle *ABC*, and join *HB*, *HD*, *HG*. And because from *H* three equal lines are drawn to the circumference *BDF*, *H* is the centre of *BDF*. Therefore the same point is the centre of two circles which intersect, which is impossible

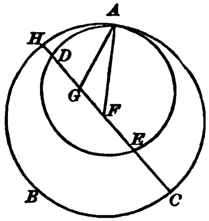
(III. 5). Therefore the two circles cannot cut in more than two points.

Proposition 10.

Theorem.—If two circles touch each other internally, the straight line joining their centres, if produced, will pass through the point of contact.

Let *ABC*, *ADE* touch each other internally at *A*, and let *F* and *G* be their centres; the line *FG* will pass through *A*.

For, if not, let it fall otherwise, as *CH*. Join *AF*, *AG*; and because *G* is the centre of *ADE*, *DE* is a diameter, and *F* a point in it. Hence (III. 7) *FD* is greater than *FA*; but *FA* is equal to *FH*; hence *FD* is greater than *FH*; which is impossible. Hence the line joining *FG* will not pass otherwise than through *A*.



Corollary 1.—If two circles touch each other internally, the distance between their centres will be equal to the difference of their radii. And, conversely, if the distance between the centres be equal to the difference of the radii, the two circles will touch each other internally.

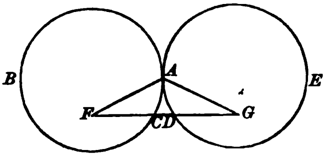
Corollary 2.—Two circles cannot touch each other internally in more than one point.

For, if possible, the line joining the points of contact would be a common chord, and the line perpendicular to this at the middle point would be a diameter (III. 1, Cor.) to both circles, and therefore pass through the point of contact, which is impossible.

Proposition 11.

Theorem.*—If two circles touch each other externally, the straight line joining the centres will pass through the point of contact.*

Let the two circles, ABC, ADE, touch each other externally at A, and let F and G be their centres; the straight line FG will pass through A.

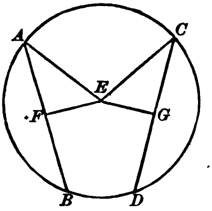


If not, let it pass otherwise, as $FCDG$, and join FA, GA. Now FA, FC, being radii of ABC, are equal; also, GA, GD, being radii of ADE, are equal. Hence, FA, AG are together equal to FC, GD; therefore FG is greater than FA, AG; which is impossible. Therefore the line between the centres will not pass otherwise than through A.

Corollary.*—If two circles touch each other externally, the distance between their centres will be equal to the sum of their radii. And, conversely, if the distance between the centres be equal to the sum of the radii, the two circles will touch externally.*

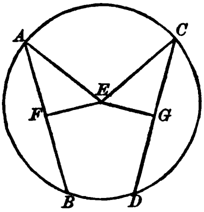
Proposition 12.

Theorem.*—Equal chords are equally distant from the centre; and those which are equally distant from the centre are equal to one another.*



Let the chords AB, CD in a circle be equal to each other; they will be equally distant from the centre.

Take E, the centre of $ABDC$, and draw EF, EG at right angles to AB, CD; join AE, EC. Then, because EF, EG from the centre are at right angles to AB, CD, they bisect (III. 3) them; hence AB is double AF, and CD double CG; therefore AF is equal to CG.



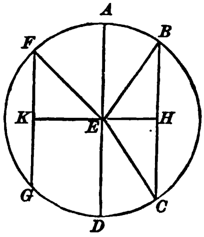
Now, in the right-angled triangles AEF, CEG, because AF is equal to CG, and AE to EC, therefore EF is equal (I. 42, Cor. 3) to EG; therefore AB, CD are equally distant from the centre (III. Def. 3).

Also, if we suppose EF equal to EG, by a similar method we may prove AF is equal to CG; hence AB is equal to CD.

Proposition 13.

***Theorem.**—The diameter is the greatest straight line in a circle; and of all others, that which is nearer the centre is greater than one more remote; and the greater is nearer the centre than the less.*

Let $ABCD$ be a circle, and AD the diameter; and let BC be nearer the centre than FG; then will AD be greater than BC, and BC than FG.



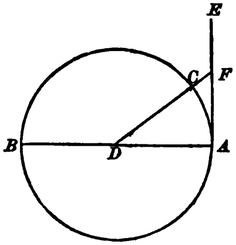
From the centre, E, draw EK, EH at right angles to FG, BC, and join EB, EC, EF; and because AE is equal to EB, and ED to EC, AD is equal to BE and EC; but BE and EC are greater than BC; hence AD is greater than BC.

Because BC is nearer the centre than FG, EH is less (III. Def. 4) than EK. But FG is double FK, and BC double BH; also the squares of EH, HB are equal to the squares EK, KF, both being equal (I. 42) to the square of the radius; but the square of EH is less than the square of EK, hence the square of BH is greater than the square of FK, and BH than FK, and therefore BC is greater than FG.

Next let BC be greater than FG. By a similar proof, EH may be shown less than EK; that is, BC is nearer the centre than FG.

Proposition 14.

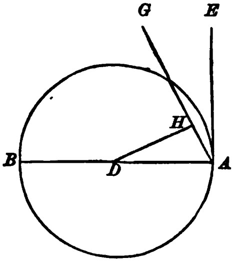
Theorem.—The straight line drawn at right angles to the diameter of a circle from the extremity of it, falls without the circle; and no straight line can be drawn between it and the circumference from the extremity of the diameter which does not cut the circle.



Let ABC be a circle, AB its diameter, and AE drawn at right angles to it; AE will fall without the circle.

In AE take any point F and join DF, and let DF meet the circumference in C.

Because DAF is a right angle, it is greater than DFA (I. 30); hence the opposite side DF is greater (I. 21) than DA or DC. Hence F is without the circle, and the same may be proved of any point of the line AE; therefore AE is without the circle.



Again between AE and the circumference no straight line can be drawn which does not cut the circle.

For, draw AG, and DH at right angles to it; then DHA being a right angle, it is greater than DAH, and DA is greater than DH. Hence H is within the circle, and AG cuts the circle.

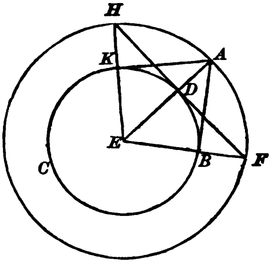
Corollary 1.—The straight line drawn at right angles to the diameter from its extremity touches the circle, and touches it only in one point. Also, there can but one straight line touch a circle in a given point.

Corollary 2.—A tangent is at right angles to the radius *through the point of contact. Also the line drawn from the point of contact to the centre is at right angles to the tangent.*

Corollary 3.—*If a line touch a circle, and from the point of contact a perpendicular be drawn, the centre of the circle is in that line.*

Proposition 15.

Problem.—*To draw a straight line from a given point, tangent to a given circle.*



1. Let *A* be the point and *BCD* the circle; it is required to draw a straight line from *A* tangent to *BCD*.

Find *E* the centre of the circle; join *AE*, and with centre *E* and radius *EA* describe the circle *AFH*; from *D* draw *DF* at right angles to *AE*; join *EF* and *AB*; *AB* touches the circle.

Because in the two triangles *AEB*, *FED*, *AE*, *EB* and *AEB* are equal to *FE*, *ED* and *FED*, each to each, therefore (I. 7) *EBA* is equal to *EDF*; but *EDF* is a right angle, therefore *EBA* is a right angle and *AB* is a tangent (III. 14, Cor. 1).

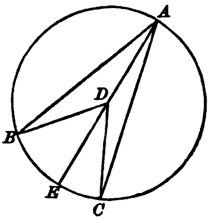
2. Let *D* be the point. Draw *DF* at right angles to *ED*. *DF* is the required tangent.

Scholium.—From *A* there can be drawn two equal tangents to *BCD*. For, produce *FD* to *H*; join *EH* and *AK*. Then *AKE*, *ABE* are equal right-angled triangles; hence, *AK* is equal to *AB*.

Corollary.—*A line from a point without a circle to the centre bisects the angle formed by two tangents from this point.*

Proposition 16.

Theorem.*—The angle at the centre of a circle is double of the angle at the circumference upon the same base; that is, upon the same part of the circumference.*



Let ABC be a circle, BDC the angle at the centre, and BAC the angle at the circumference upon the same arc BC; BDC is double of BAC.

Let D fall within BAC. Join AD and produce it to E.

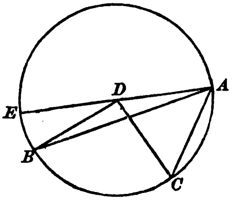
Because $DB = DA$, therefore (I. 13) $DAB = DBA$ and $DAB + DBA = 2DAB$.

Now (I. 30) $BDE = DAB + DBA$;

$\therefore \quad BDE = 2BAE.$

Similarly, $CDE = 2CAE$;

$\therefore \quad BDC = 2BAC.$

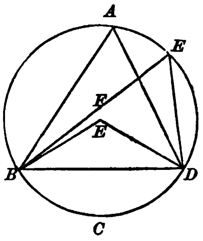


Let D fall outside of BAC. Then (last case) $EDB = 2EAB$ and $EDC = 2EAC$, subtracting (Ax. 3) $BDC = 2BAC$.

Proposition 17.

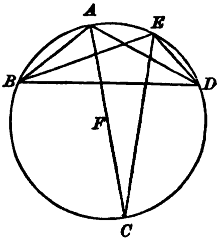
Theorem.*—The angles in the same segment of a circle are equal to one another.*

Let $ABCD$ be a circle, and BAD, BED angles in the same segment $BAED$; the angles BAD, BED are equal to each other. Take F, the centre of the circle, $ABCD$. And first, let the segment $BAED$ be greater than a semicircle, and join BF, FD; and because the angle BFD is at the centre, and the angles BAD, BED at the circumference, both standing on the same arc BCD, therefore the angle BFD is



double (III. 16) of the angles BAD, BED; therefore the angle BAD is equal to the angle BED (Ax. 7).

But, if the segment $BAED$ be not greater than a semicircle, let BAD, BED be angles in it; these also are equal to each other. Draw AF to the centre and produce it to C, and join CE; therefore the segment $BADC$ is greater than a semicircle; and the angles in it, BAC, BEC, are equal by the first case; for the same reason, because $CBED$ is greater than a semicircle, the angles CAD, CED are equal; therefore the whole angle BAD is equal to the whole angle BED.



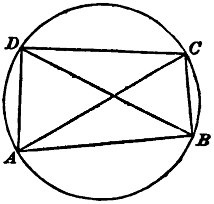
Corollary.—The angles in a circle standing on the same arc are equal to one another.

Proposition 18.

Theorem.—The opposite angles of any quadrilateral figure described in a circle are together equal to two right angles.

Let $ABCD$ be a quadrilateral figure in the circle $ABCD$; any two of its opposite angles are together equal to two right angles.

Join AC, BD. The angle CAB is equal (III. 17) to the angle CDB, because they are in the same segment $BADC$, and the angle ACB is equal to the angle ADB, because they are in the same segment $ADCB$; therefore the whole angle ADC is equal to the angles CAB, ACB; to each of these equals add the angle ABC, and the angles ABC, ADC are equal to the angles ABC, ACB, CAB. But ABC, ACB, CAB are equal to two right angles (I. 30); therefore



also the angles ABC, ADC are equal to two right angles. In the same manner, the angles BAD, DCB may be shown to be equal to two right angles.

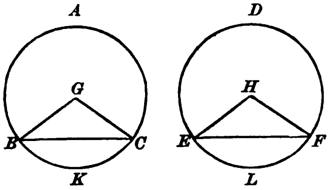
Corollary 1.—*If any side of a quadrilateral be produced, the exterior angle will be equal to the opposite interior angle.*

Corollary 2.—*A quadrilateral, of which the opposite angles are not equal to two right angles, cannot be inscribed in a circle.*

Proposition 19.

Theorem.—*In equal circles equal angles at the centre stand on equal arcs; and, conversely, equal arcs are subtended by equal angles at the centre.*

Let ABC, DEF be equal circles, and BGC, EHF equal angles at their centres; the arc BKC is equal to the arc ELF.



For let the circle ABC be applied to the circle DEF, so that B shall be on E and BG on EH; and because the circles are equal their radii are equal, and the point G will be on H; also because the angle BGC is equal to the angle EHF, the point C will be on F. Now, because the circumferences coincide and the points B and C coincide with E and F, the arc BKC must coincide with the arc ELF and be equal to it.

Conversely, if BKC be equal to ELF, the angle BGC is equal to the angle EHF.

For let the circle ABC be applied as before, with B on E, then because the arc BKC is equal to the arc ELF, C will coincide with F; also because the circles are equal G will coincide with *H*. Hence the angle *BGC* is equal to the angle *EHF*.

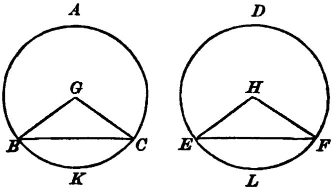
***Corollary.**—In equal circles, equal angles at the circumference stand on equal arcs, and conversely.*

For the angles at the circumference are halves of the angles at the centre.

Proposition 20.

***Theorem.**—In equal circles equal straight lines cut off equal arcs, and, conversely, equal arcs are subtended by equal straight lines.*

Let *ABC*, *DEF* be equal circles, and let *BC*, *EF* be equal straight lines in them, then the arc *BKC* is equal to the arc *ELF*.



Take (III. 1) *G* and *H*, the centres of the circles, and join *BG*, *GC*, *EH*, *HF*; and because the circles are equal *BG*, *GC* are equal to *EH*, *HF*, and *BC* is equal to *EF*; therefore the angle *BGC* is equal (I. 4) to the angle *EHF*. But equal angles stand on equal arcs (III. 19); therefore, the arc *BKC* is equal to the arc *ELF*.

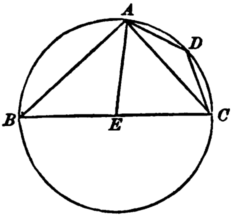
Conversely, if *BKC* be equal to *ELF*, *BC* is equal to *EF*.

For because *BKC* is equal to *ELF*, the angle *BGC* is equal (III. 19) to the angle *EHF*, and *BG*, *GC* are equal to *EH* *HF*; therefore *BC* is equal (I. 7) to *EF*.

Proposition 21.

***Theorem.**—The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle, is less than a right angle; and the angle in a segment less than a semicircle, is greater than a right angle.*

Let *ABCD* be a circle of which the diameter is *BC* and centre *E*; draw *CA*, dividing the circle into segments *ABC*, *ADC*, and join *BA*, *AD*, *DC*; the angle in the semicircle *BAC* is a right angle; the angle in the segment *ABC* is less than a right angle; and the angle in the segment *ADC* is greater than a right angle.



Join *AE*; and because *BE* is equal to *EA*, the angle *EAB* is equal (I. 13) to the angle *EBA*; also because *EA* is equal to *EC*, the angle *EAC* is equal to *ECA*; therefore the whole angle *BAC* is equal to the two angles *ABC* and *ACB*, and consequently is one half the sum of the angles of the triangle *ABC*; therefore (I. 30) the angle *BAC* in a semicircle is a right angle.

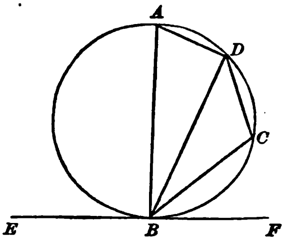
And because the two angles *ABC*, *BAC* of the triangle *ABC* are together less (I. 19) than two right angles, and *BAC* is a right angle, *ABC* must be less than a right angle.

Also, because *ABCD* is a quadrilateral figure in a circle, any two of its opposite angles, as *ABC*, *ADC*, are equal (III. 18) to two right angles; and *ABC* is less than a right angle; wherefore the other, *ADC*, is greater than a right angle.

Proposition 22.

Theorem.—*If a straight line touch a circle, and from the point of contact a straight line be drawn cutting the circle, the angles made by this line with the line which touches the circle, will be equal to the angles in the alternate segments of the circle.*

Let the straight line *EF* touch the circle *ABCD* in *B*, and from the point *B* let the straight line *BD* be drawn cutting the circle; the angle *FBD* is equal to the angle which is in the segment *DAB*, and the angle *DBE* to the angle in the segment *BCD*.



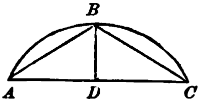
From the point B draw (I. 11) BA at right angles to EF, and take any point C in the arc BD, and join AD, DC, CB. Now (III. 14, Cor. 3), AB is a diameter, and the angle ADB in a semicircle is a right angle (III. 21), and consequently the other two angles, DAB and DBA, are equal (I. 30) to a right angle, and therefore equal to ABF. Take from these equals the common angle ABD, and there will remain the DBF, equal to the angle BAD, which is in the alternate segment of the circle.

And because $ABCD$ is a quadrilateral figure in a circle, the opposite angles BAD, BCD are equal (III. 18) to two right angles; therefore the angles DBF, DBE, being likewise equal (I. 15) to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD; therefore the remaining angle DBE is equal to the angle BCD in the alternate segment of the circle.

Proposition 23.

Problem.—*To bisect a given arc.*

Let ABC be a given arc; it is required to bisect it.



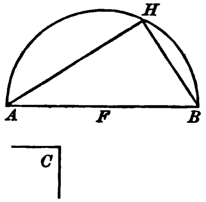
Join AC, and bisect (I. 10) it in D. From the point D draw DB at right angles to AC, and join AB, BC.

Because in the triangles ADB, CDB, AD, DB and the angle ADB of one, are respectively equal to CD, DB, and CDB of the other, AB is equal (I. 7) to BC. But equal lines cut off equal arcs; hence the arcs AB, BC are equal, and ABC is bisected.

Proposition 24.

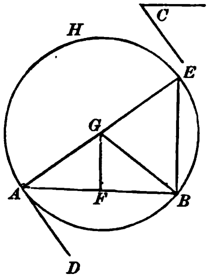
Problem.—*Upon a given straight line to describe a segment of a circle, containing an angle equal to a given rectilineal angle.*

Let *AB* be the given straight line, and the angle at *C* the given rectilineal angle; it is required to describe upon *AB* a segment of a circle containing an angle equal to *C*.



First, let *C* be a right angle; bisect (I. 10) *AB* in *F*, and from the centre, *F*, at the distance *FB*, describe the semicircle *AHB*; the angle *AHB* being in a semicircle is (III. 21) equal to the right angle at *C*.

But if the angle at *C* be not a right angle, at the point *A* in the straight line *AB*, make (I. 5) the angle *BAD* equal to the angle at *C*, and from the point *A* draw (I. 11) *AE* at right angles to *AD*; bisect (I. 10) *AB* in *F*, and from *F* draw (I. 11) *FG* at right angles to *AB*, and join *GB*; then in the triangles *AFG*, *BFG*, the two sides *AF*, *FG* are equal to the two *BF*, *FG*; and the angle *AFG* is also equal to the angle *BFG*; therefore *AG* is equal (I. 7) to *GB*; and the circle described from the centre *G*, at the distance *GA*, will pass through the point *B*; let this be the circle *AHB*; and because from the point *A*, the extremity of the diameter *AE*, *AD* is drawn at right angles to *AE*, therefore *AD* (III. 14, Cor. 1) touches the circle; and because *AB*, drawn from the point of contact, *A*, cuts the circle, the angle *DAB* is equal to the angle in the alternate segment *AHB* (III. 22); but the angle *DAB* is equal to the angle *C*, therefore also the angle *C* is equal to the angle in the segment

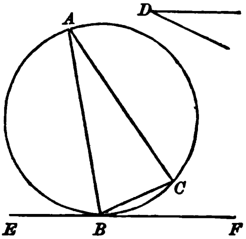


AHB; wherefore upon *AB* the segment *AHB* is described, which contains an angle equal to *C*.

Proposition 25.

***Problem.**—To cut off a segment from a given circle which shall contain an angle equal to a given rectilineal angle.*

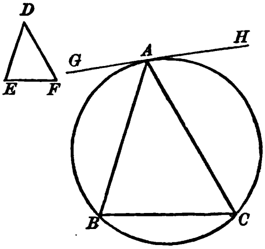
Let *ABC* be the given circle, and *D* the given rectilineal angle; it is required to cut off a segment from the circle *ABC*, which shall contain an angle equal to the angle at *D*.



Draw (III. 15) *EF*, touching the circle *ABC* in the point *B*, and make (I. 5) the angle *FBC* equal to the angle *D*. Therefore because *EF* touches the circle *ABC*, and *BC* is drawn from the point of contact *B*, the angle *FBC* is equal (III. 22) to the angle in the alternate segment *BAC*; but *FBC* is equal to *D*; therefore the angle in the segment *BAC* is equal to *D*; wherefore the segment *BAC* is cut off the circle *ABC*, containing an angle equal to *D*.

Proposition 26.

***Problem.**—In a given circle to inscribe a triangle equiangular to a given triangle.*

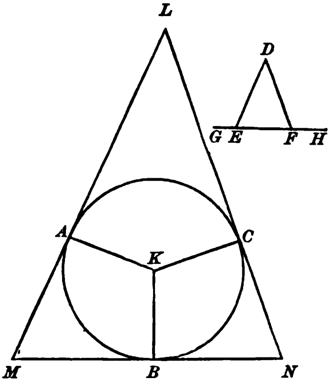


Let *ABC* be the given circle, and *DEF* the given triangle; it is required to inscribe in the circle *ABC* a triangle equiangular to *DEF*.

Draw (III. 15) *GAH* touching the circle in the point *A*, and make (I. 5) the angle *HAC* equal to the angle *DEF*, and the angle *GAB* equal to the angle *DFE*, and join *BC*. The angle *HAC* is equal (III. 22) to the angle *ABC* in the alternate segment of the circle. But *HAC* is equal to the angle *DEF*; for the same reason the angle *ACB* is equal to the angle *DFE*; therefore the remaining angle *BAC* is equal (I. 30, Cor. 4) to the remaining angle *EDF*; wherefore the triangle *ABC* is equiangular to the triangle *DEF*, and is inscribed in the circle *ABC*.

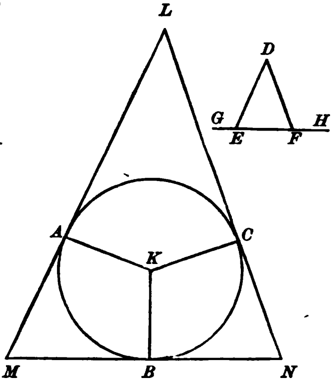
Proposition 27.

***Problem.**—About a given circle to describe a triangle equiangular to a given triangle.*



Let *ABC* be the given circle, and *DEF* the given triangle; it is required to describe a triangle about the circle *ABC* equiangular to *DEF*.

Produce *EF* both ways to the points *G*, *H*, and find (III. 1) the centre *K*, of the circle *ABC*, and from it draw any straight line *KB*; make (I. 5) the angle *BKA* equal to the angle *DEG*, and the angle *BKC* equal to the angle *DFH*; and through the points *A*, *B*, *C* draw the straight lines *LAM*, *MBN*, *NCL* touching (III. 15) the circle *ABC*; therefore because *LM*, *MN*, *NL* touch the circle *ABC*, and *KA*, *KB*, *KC* are drawn to the centre, the angles at the points *A*, *B*, *C* are right (III. 14, Cor. 2) angles, and the four angles of the quadrilateral figure *AMBK* are equal to four right angles, because it can be divided into two triangles; and because two of them, *KAM*, *KBM*, are right angles, the other two, *AKB*, *AMB*, are equal to two right angles. But the angles *DEG*, *DEF* are likewise equal (I. 15) to two right angles; therefore the angles *AKB*, *AMB* are equal to the angles *DEG*, *DEF*, of which *AKB* is equal to *DEG*; wherefore the remaining angle *AMB* is equal to the remaining angle *DEF*. In like manner the angle *LNM* may be proved equal to the angle *DFE*; and therefore the remaining angle *MLN* is equal (I. 30, Cor. 4) to the remaining angle *EDF*; wherefore the triangle *LMN* is equiangular to the triangle *DEF*, and it is described about the circle *ABC*.

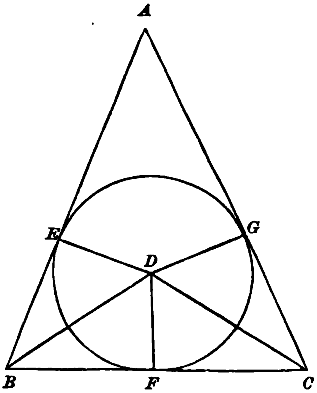


Proposition 28.

Problem.—To inscribe a circle in a given triangle.

Let the given triangle be *ABC*; it is required to inscribe a circle in *ABC*.

Bisect (I. 9) the angles *ABC*, *BCA* by the straight lines *BD*, *CD*, meeting each other in the point *D*, from which draw (I. 12) *DE*, *DF*, *DG*, perpendiculars to *AB*, *BC*, *CA*. Then, because the angle *EBD* is equal to the angle *FBD*, and the right angles *BED*, *BFD* are equal, the two triangles *EBD*, *FBD* have two angles of the one equal to two angles of the other; and the side *BD* common to both; therefore their other sides are equal (I. 24); wherefore *DE* is equal to *DF*. For the



same reason, DG is equal to DF; therefore DE, DF, DG are equal to one another, and the circle described from the centre D, at the distance of any of them, will pass through the extremities of the other two, and will touch the straight lines AB, BC, CA (III. 14, Cor.). Therefore the circle EFG is inscribed in the triangle ABC.

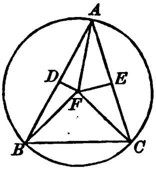
***Corollary.**—From this it is manifest that the three lines bisecting the angles of a triangle intersect in the same point.*

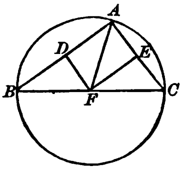
Proposition 29.

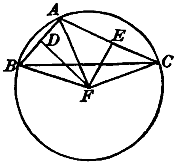
***Problem.**—To describe a circle about a given triangle.*

Let the given triangle be ABC; it is required to describe a circle about ABC.

Bisect (I. 12) AB, AC in the points D, E, and from these points draw DF, EF at right angles (I. 11) to AB, AC; DF, EF produced will meet each other; for if they do not meet they are parallel, wherefore AB, AC, which are at right angles to them, are parallel, which is absurd. Let them meet in F, and join FA, FB, FC; then, because AD is equal to BD, and DF common, and at right angles to AB, the base AF is equal (I. 7) to the base FB. In like manner CF is equal to FA; therefore FA, FB, FC are equal to one another; wherefore the circle described from the centre F, at the distance of one of them, will pass through the extremities of the other two, and be described about the triangle ABC.







***Corollary 1.**—From this it is evident that the three lines at*

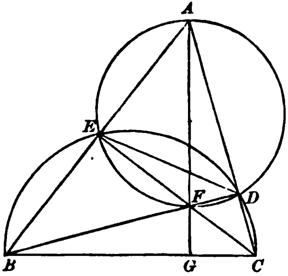
right angles to the sides of a triangle, from their middle points, intersect in the same point.

Corollary 2.—If an arc of a circle be given, we can find its centre by taking three points in it, joining them, and finding the centre of the circumscribed triangle.

Corollary 3.—If the triangle be right-angled, the centre is on the hypotenuse; also, if a circle be described on the hypotenuse, it will pass through the right angle.

Proposition 30.

Theorem.—The perpendiculars to the three sides of a triangle from the opposite angles intersect in the same point.



Let ABC be a triangle, and BD, CE the perpendiculars on AC, AB respectively. Let them intersect at F; join AF, and produce it, if necessary, to meet BC in G; then AGB is a right angle.

Join ED; upon AF and BC as diameters describe circles; because the angle AEF is a right angle, the circle on AF will pass through the point E (III. 29, Cor. 3); for the same reason it will pass through the point D. Similarly, the circle on BC will pass through E and D.

Because the angles EAF, EDF are in the same segment $EADF$, they are equal to each other; and because the angles EDB, ECB are in the same segment $EDCB$, they are equal to each other; therefore the angles EAF and ECB are equal to each other. Then, because in the triangles BAG, BEC the angle B is common, and the angles BAG, BCE equal to each other, therefore the remaining angles BEC, BGA are equal to each other; but BEC is a right angle; therefore BGA is a right angle, and AG is perpendicular to BC.

EXERCISES.

1. To describe a circle that shall pass through three given points not in the same line (III. 1).

2. To describe a circle that shall pass through two given points and have a given radius, which must not be less than half the distance between the points.

3. To describe a circle that shall be tangent to two intersecting lines and have a given radius.

4. To describe three equal circles tangent to one another.

Construct an equilateral triangle; make the three angles the centres, and half the sides, the radii of the circles.

5. Show, in the last example, the common tangent to two of the circles will pass through the centre of the other.

6. To draw, parallel to a given straight line, a tangent to a given circle.

7. To draw, perpendicular to a given straight line, a tangent to a given circle.

8. A quadrilateral is described so that its sides touch a circle. One pair of its opposite sides is equal to the other (III. 15, Sch.).

9. No parallelogram but an equilateral one can be described about a circle (III. 15, Sch.).

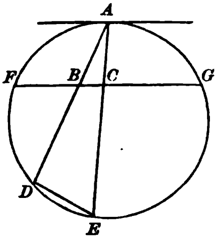
10. No parallelogram but a rectangular one can be inscribed in a circle (III. 21).

11. A triangle is inscribed in a circle; the three angles in the segments exterior to the triangle, are equal to four right angles (III. 18).

12. A quadrilateral is inscribed in a circle; the four angles in the segments exterior to the quadrilateral, are equal to six right angles (III. 18).

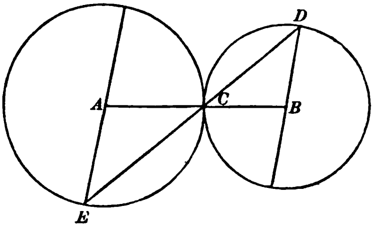
13. The straight lines in a circle which join the extremities of parallel chords, are equal to each other.

14. If the chord *FG* be drawn parallel to the tangent at *A*, the triangles *ABC*, *ADE* shall have equal angles (III. 22).

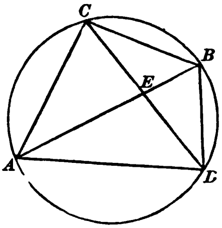


15. If two circles touch each other externally, the straight lines joining the opposite ends of two parallel chords shall pass through the point of contact.

Join *AB*; it will pass through the point of contact; draw *BD* any direction; join *DC*, and produce it to meet the other circumference in *E*; join *EA*. Now prove *EA* is parallel to *DB*.

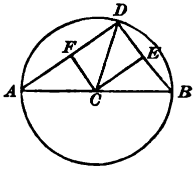


16. If any two chords, *AB* and *CD*, cut each other in a circle, and their extremities be joined, the triangles *AED*, *CEB*, as also *AEC*, *BED*, will have equal angles (III. 17).



17. If any number of right-angled triangles are described on the same hypotenuse, their vertices are in the circumference of the same circle.

Let *ABD* be a triangle, and *AB* the hypotenuse. Bisect the hypotenuse, and join the point of bisection with the opposite angle; draw *CF*, *CE* parallel to the sides. *EF* is a rectangle, because *D* is a right angle. In the triangles *ACF*, *CEB*, *CF* = *EB*.

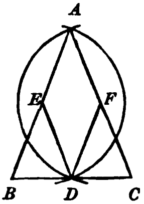


$\therefore \qquad DE = EB, \qquad DF = AF.$

Show now that the circle on *AB* will pass through *D*.

18. The circles described on the equal sides of an isosceles triangle as diameters will intersect at the middle point of the base.

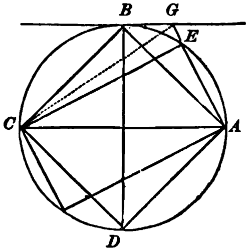
Join the middle points *FD*, *ED*.



See the 18th Example, Book I., to prove that *ED* is parallel to *AC*, and *DF* to *AB*. $\therefore$ *AD* is a parallelogram. Hence the circles described with *E* and *F* as centres, and radii *EA*, *FA*, will pass through *D*.

19. The greatest rectangle that can be inscribed in a circle is a square.

We want to prove that *ABC*, half a square, is greater than *AEC*, half of any other rectangle. Draw *BG* parallel to *AC*, and let *AE* meet it in *G*.



20. An exterior angle of an inscribed quadrilateral is 30°. How many interior angles are known?

21. The angle between a tangent and diameter is bisected. How many degrees in the angles in the two segments formed?

22. A tangent makes an angle of 60° with a chord. What portion of the circumference does the chord cut off?

23. If circles be described in and about an equilateral triangle, they have the same centre.

24. If equilateral triangles be described in and about a circle, the area of one will be four times the area of the other.

Pass the sides of one through the angles of the other.

25. Two tangents from the same point outside a circle make equal angles with the chord joining the points of contact.

26. If the three points in which the inscribed circle meets the sides of a triangle be joined, the triangle formed will be acute-angled.

27. If tangents be drawn through the extremities of two diameters of a circle, a rhombus is formed.

28. Find the centre of a circle, cutting off equal chords from the sides of a triangle.

It is the centre of the inscribed circle. Prove it.

BOOK IV.

RATIOS.

DEFINITIONS.

1. **Ratio** is the relation with respect to magnitude, which one quantity bears to another of the same kind, and is the quotient arising from dividing the first by the second.

The ratio of A to B is $\frac{A}{B}$; or, as it is usually written, $A : B$; the ratio of 2 to 4, is $\frac{2}{4}$ or $\frac{1}{2}$.

2. A **proportion** expresses the equality of two or more ratios.

If the ratio of A to B be the same as the ratio of C to D, these quantities form a proportion; thus, $\frac{A}{B}=\frac{C}{D}$, or $A : B :: C : D$. Hence a proportion is an equation with both members fractional.

3. Ratio may be *direct* or *inverse.*

If x and y be two variable quantities, such that, as x increases, y increases at the same rate (that is, as x doubles, y doubles, etc.), x and y have a direct ratio to each other. If, as x increases, y diminishes proportionally (that is, as x doubles, y becomes one-half its former value, etc.), x and y have an inverse ratio to each other. Since, in an inverse ratio, x is proportional to the reciprocal of y, an inverse ratio is frequently called a *reciprocal* ratio.

4. The first terms of the ratios (that is, the first and third terms of the proportion) are called **antecedents,** and the second and fourth, **consequents.**

5. The first and fourth terms are called **extremes,** and the second and third, **means.**

6. Quantities are in **continued proportion** when a consequent of one ratio is the same as the antecedent of the next.

Thus, $A : B :: B : C :: C : D$, etc.

7. The second quantity is then said to be a **mean proportional** between the first and third; and the third is a **third proportional** to the first and second.

8. In a proportion the fourth term is said to be a **fourth proportional** to the other three taken in order.

9. Quantities are in proportion **alternately,** when antecedent is compared with antecedent, and consequent with consequent.

If $A : B :: C : D$, then, alternately, $A : C :: B : D$.

10. Quantities are in proportion **inversely,** when antecedent is made consequent, and consequent, antecedent.

If $A : B :: C : D$, then, inversely, $B : A :: D : C$.

11. Quantities are in proportion by **composition,** when the sum of antecedent and consequent is compared with either antecedent or consequent.

If $A : B :: C : D$, then, by composition,

$$A+B : A \text{ or } B :: C+D : C \text{ or } D.$$

12. Quantities are in proportion by **division,** when the difference of antecedent and consequent is compared with either antecedent or consequent.

If $A : B :: C : D$, then, by division,

$$A-B : A \text{ or } B :: C-D : C \text{ or } D.$$

Proposition 1.

Theorem.—*If four quantities be in proportion, the product of the extremes is equal to the product of the means.*

Let $A : B :: C : D,$

then $A \times D = B \times C.$

For, because $\frac{A}{B} = \frac{C}{D}$, multiplying by $B \times D$, we have

$$A \times D = B \times C.$$

Corollary 1.—*If three quantities be in continued proportion, the product of the extremes is equal to the square of the mean.*

If $A : B :: B : C,$

then, by this proposition,

$$A \times C = B \times B = B^2.$$

Corollary 2.—*A mean proportional between two quantities is the square root of their product; for* $B = \sqrt{A \times C}.$

Scholium.—When we speak of the product of two quantities, one of them at least must be a number. We cannot have the product of two lines, or of two surfaces, or of two solids, using these terms in their geometrical sense. When we, for convenience, use the expression *product of two lines,* we mean the number of units in the length of one, multiplied by the number of units of the same kind in the length of the other. Expressed in this way, lines become numerical quantities, and may be used as factors. If A and B be two lines, and C the common unit of measure, and if A contain C, m times, and B contain C, n times, then evidently $\frac{A}{B} = \frac{mC}{nC} = \frac{m}{n}$. Hence, when we have the ratio of two lines, we may substitute the ratio of their numerical measures. Hence, the three succeeding propositions are true for lines as well as for numbers; and in the same way they may be shown true for surfaces and solids. Any of the other propositions of this book, which we afterward apply to geometrical magnitudes, we will prove by methods which do not involve the multiplication or division of two of these magnitudes.

Proposition 2.

Theorem.*—If the product of two quantities be equal to the product of two others, these four quantities may form a proportion, one set being taken for the extremes and the other for the means.*

If $A \times D = B \times C$,

then $A : B :: C : D$.

For, because $A \times D = B \times C$

Dividing both sides by $B \times D$, we have $\frac{A}{B} = \frac{C}{D}$,

or $A : B :: C : D$.

Scholium.—From this it is manifest that we can form a number of proportions from the equation $A \times D = B \times C$. The only limitation being, that if A be made a mean or an extreme, D must be the other mean or extreme. Thus we might have

$$A : C :: B : D,$$
$$B : A :: D : C,$$
$$C : A :: D : B, \text{ etc.}$$

Proposition 3.

Theorem.*—Equimultiples of two quantities are proportional to the quantities.*

Let A and B be two quantities, and mA and mB equimultiples of them.

Then $A : B :: mA : mB$.

For $\frac{A}{B} = \frac{mA}{mB}$, or $A : B :: mA : mB$.

Corollary.—*Equal measures of two quantities are proportional to the quantities.*

This may be proved as the above by supposing m to be fractional.

Proposition 4.

Theorem.—*If equimultiples be taken of the first and second terms of a proportion, and also of the third and fourth, the resulting terms are proportional.*

If $A : B :: C : D,$

then $mA : mB :: nC : nD.$

For, because $\frac{A}{B}=\frac{C}{D}, \frac{mA}{mB}=\frac{nC}{nD},$

or, $mA : mB :: nC : nD.$

Proposition 5.

Theorem.—*If equimultiples be taken of the first and third terms of a proportion, and also of the second and fourth, the resulting terms are proportional.*

If $A : B :: C : D,$

then $mA : nB :: mC : nD.$

For, because $\frac{A}{B}=\frac{C}{D}$, multiplying by $\frac{m}{n}$,

$$\frac{mA}{nB}=\frac{mC}{nD},$$

or, $mA : nB :: mC : nD.$

Proposition 6.

Theorem.—*If four quantities be in proportion, if the first be any multiple of the second, the third is the same multiple of the fourth.*

If $A : B :: C : D$, and $A = mB$,
then $C = mD$.

For, because $\frac{A}{B} = \frac{C}{D}$, and $A = mB$,

therefore, $\frac{mB}{B} = \frac{C}{D}$, or $\frac{C}{D} = m$, whence $C = mD$.

Proposition 7.

Theorem.—*If four quantities of the same kind be in proportion, they are in proportion when taken alternately.*

If $A : B :: C : D$,
then $A : C :: B : D$.
From (IV. 1) $A \times D = B \times C$,
therefore (IV. 2), $A : C :: B : D$.

Proposition 8.

Theorem.—*If four quantities be in proportion, they are in proportion when taken inversely.*

If $A : B :: C : D$,
then $B : A :: D : C$.
From (IV. 1) $A \times D = B \times C$,
therefore (IV. 2), $B : A :: D : C$.

Proposition 9.

Theorem.—*If four quantities be in proportion, they are in proportion by composition.*

If $A : B :: C : D$,
then $A+B : B :: C+D : D$,
and $A+B : A :: C+D : C$.

For, because $\frac{A}{B}=\frac{C}{D}$, adding 1 to both members,

$$\frac{A}{B}+1=\frac{C}{D}+1, \text{ or } \frac{A+B}{B}=\frac{C+D}{D},$$

or, $A+B : B :: C+D : C.$

Also (IV. 8), $\frac{B}{A}=\frac{D}{C}$, whence, as before,

$$A+B : A :: C+D : C.$$

Proposition 10.

Theorem.—If four quantities be in proportion, they are in proportion by division.

If $A : B :: C : D,$

then $A-B : B :: C-D : D,$

and $A-B : A :: C-D : C.$

For, because $\frac{A}{B}=\frac{C}{D}$, subtracting 1 from both members,

$$\frac{A}{B}-1=\frac{C}{D}-1, \text{ or } \frac{A-B}{B}=\frac{C-D}{D},$$

or, $A-B : B :: C-D : D.$

Also, because $\frac{B}{A}=\frac{D}{C}$, we have

$$A-B : A :: C-D : C.$$

Proposition 11.

Theorem.—If four quantities be in proportion, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

If $A : B :: C : D,$

then $A+B : A-B :: C+D : C-D.$

From (IV. 9) $$\frac{A+B}{B}=\frac{C+D}{D}.$$

From (IV. 10) $$\frac{A-B}{B}=\frac{C-D}{D}.$$

Dividing one equation by the other, and striking out (IV. 3) the common terms, B and D, and we have

$$\frac{A+B}{A-B}=\frac{C+D}{C-D},$$

or, $$A+B : A-B :: C+D : C-D.$$

Proposition 12.

Theorem.*—The products of the corresponding terms of two numerical proportions are in proportion.*

If $$A : B :: C : D,$$
and $$E : F :: G : H,$$
then $$A.E : B.F :: C.G : D.H.$$

For, because $$\frac{A}{B}=\frac{C}{D}, \text{and } \frac{E}{F}=\frac{G}{H},$$

multiplying the equations together,

$$\frac{A.E}{B.F}=\frac{C.G}{D.H},$$

or, $$A.E : B.F :: C.G : D.H.$$

Corollary 1.*—The continued products of the corresponding terms of any number of proportions are in proportion.*

Corollary 2.*—If four quantities be in proportion, like powers of them are in proportion.*

For, if in last corollary all the corresponding terms be the same, the products would be like powers of the terms.

Proposition 13.

Theorem.—*If two proportions have in each ratio an antecedent of one the same as a consequent of the other, the other terms are in proportion, antecedent remaining antecedent, and consequent, consequent.*

If $A : B :: C : D$,
and $B : E :: F : C$,
then $A : E :: F : D$.

Let the ratio of A to B or $\frac{A}{B}=m$, and $\frac{B}{E}=n$, then $A=mB$ and $B=nE$; therefore $A=mnE$ or $\frac{A}{E}=mn$. But if $A=mB$, $C=mD$ (IV. 6), and if $B=nE$, $F=nC$; therefore $F=mnD$ or $\frac{F}{D}=mn$. Hence $\frac{A}{E}=\frac{F}{D}$, or $A : E :: F : D$.

Corollary.—*From this we see that the ratio of A to E is the product of the ratios of A to B and of B to E; that is, if $\frac{A}{B}=m$ and $\frac{B}{E}=n$, then $\frac{A}{E}=mn$.*

Proposition 14.

Theorem.—*Equal quantities have the same ratio to the same quantity, and quantities which have the same ratio to the same quantity, are equal to each other.*

Let A and B be equal magnitudes, and C another.

Then $A : C :: B : C$.

Let $\frac{A}{C}=m$; then, because $A=B$, $\frac{B}{C}=m$.

Hence $\frac{A}{C}=\frac{B}{C}$, or $A : C :: B : C$.

Again, if $\frac{A}{C}=\frac{B}{C}$, let $A=mC$, then (IV. 6) $B=mC$. Hence $A=B$.

Proposition 15.

***Theorem.**—If two proportions have one ratio in each the same, the remaining terms are in proportion.*

Let $A : B :: C : D$,
and $E : F :: C : D$;
then $A : B :: E : F$.
For, because $\frac{A}{B}=\frac{C}{D}$ and $\frac{E}{F}=\frac{C}{D}$, therefore $\frac{A}{B}=\frac{E}{F}$,
or $A : B :: E : F$.

Proposition 16.

***Theorem.**—If any number of quantities be in proportion, any antecedent is to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

If $A : B :: C : D :: E . F$, etc.,
then $A : B :: A+C+E$, etc. : $B+D+F$, etc.
Let $A=mB$, then (IV. 6) $C=mD$ and $E=mF$, etc.
Adding these, we have

$$A+C+E, \text{ etc.} = m(B+D+F), \text{ etc.},$$

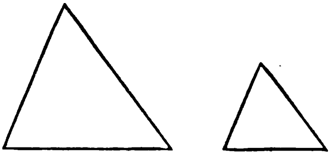
whence $\frac{A+C+E, \text{ etc.}}{B+D+F, \text{ etc.}}=m$. But $\frac{A}{B}=m$;
therefore $A : B :: A+C+E+\text{etc.} : B+D+F+\text{etc.}$

BOOK V.

SIMILAR POLYGONS.—MEASUREMENT OF POLYGONS.

DEFINITIONS.

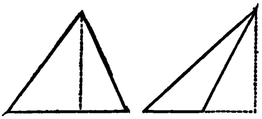
1. **Similar rectilineal figures** are those which have their several angles equal, each to each, and the sides about the equal angles proportional.



2. A straight line is cut in **extreme and mean ratio** when the whole is to the greater segment, as the greater segment is to the less.

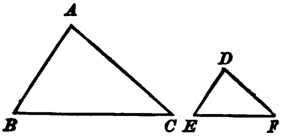
3. The **altitude** of a triangle is the straight line drawn from its vertex perpendicular to the base, or the base produced.

As any side of a triangle may be considered the base, a triangle may have three altitudes. The altitude of a parallelogram is the perpendicular distance between either pair of parallel sides.



4. The **homologous sides** of similar rectilineal figures are

those which are adjacent the equal angles; in triangles they are those which are opposite the equal angles. Thus, if $A = D$, $B = E$, and $C = F$, AB and DE are homologous sides, as also AC and DF, and BC and EF. The corresponding parts of two figures are called *homologous* whether they be lines or angles.



Proposition 1.

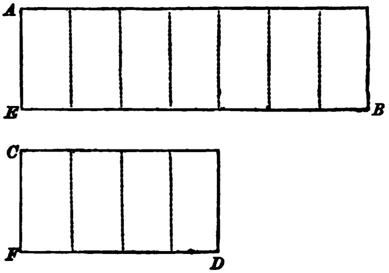
***Theorem.**—Rectangles of equal altitude are proportional to their bases.*

There are two cases: 1. Where the bases are commensurable. 2. Where the bases are incommensurable.*

1. Let AB and CD be two rectangles having equal altitudes, AE and CF;

then $$AB : CD :: EB : FD.$$

Divide the base EB into any number, as 7, equal parts, and let FD contain 4. Through the points of division draw lines parallel to AE and CF. Then AB and CD will be divided into equal (I. 34) rectangles, of which AB will contain 7 and CD 4.



$$\therefore \quad AB : CD :: 7 : 4.$$

But $$EB : FD :: 7 : 4;$$

$$\therefore \text{(IV. 15)}, \quad AB : CD :: EB : FD.$$

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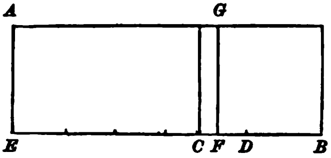
*Quantities are commensurable when they exactly contain the same unit; thus, two lines respectively 7 and 4 feet long are commensurable, but two lines respectively 7 and $4\frac{3}{4}$ feet long are incommensurable in feet.

2. They are incommensurable.



Apply the smaller rectangle AC to the larger AB, so that their bases shall be in the same line, and one angle E in each common; and if it be not true that $AB:AC::EB:EC$, suppose that

$$AB : AC :: EB : ED.$$

Lay off the unit of measure, which we take less than CD, on EB; at least one point of division, as F, will fall between C and D; draw FG parallel to AE.

Then, according to Case 1,

$$AF : AB :: EF : EB.$$

But $AB : AC :: EB : ED$;

$\therefore$ (IV. 13), $AF : AC :: EF : ED$.

But EF is less than ED, therefore AF is less than AC, which is impossible. Therefore, no other line but EC can be a fourth proportional to AB, AC, and EB.

$\therefore$ $AB : AC :: EB : EC$.

***Corollary 1.**—Parallelograms of equal altitude are proportional to their bases.*

For any parallelogram is equivalent to a rectangle having the same base and altitude (I. 33).

***Corollary 2.**—Triangles of equal altitude are proportional to their bases.*

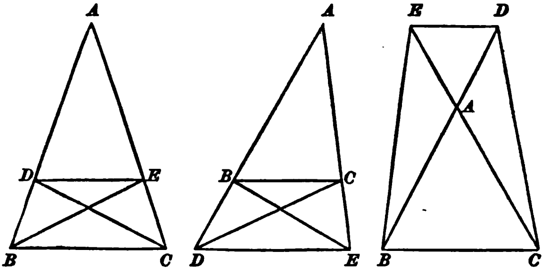
For a triangle is half a parallelogram of the same base and altitude (I. 35, Cor.).

Proposition 2.

Theorem.*—If a straight line be drawn parallel to one of the sides of a triangle, it will cut the other sides, or those sides produced proportionally; and if the sides of a triangle, or the sides produced, be cut proportionally, the straight line which joins the points of section will be parallel to the other side of the triangle.*

Let DE be drawn parallel to BC, one of the sides of the triangle ABC;

then $$BD : DA :: CE : EA.$$



Join BE, CD; then the triangle BDE is equal to the triangle CDE, because they are on the same base DE, and between the same parallels, DE, BC; and ADE is another triangle; and equal quantities have (IV. 14) the same ratio to the same quantity.

$$\therefore \quad BDE : ADE :: CDE : ADE.$$

But (V. 1, Cor. 2)

$$BDE : ADE :: BD : DA,$$

the common altitude being the perpendicular drawn from E to AB.

Similarly, $CDE : ADE :: CE : EA$;

$\therefore$ (IV. 15), $BD : DA :: CE : EA$.

Next, let $BD : DA :: CE : EA$;

then will DE be parallel to BC.

Because $BD : DA :: CE : EA$,

and (V. 1, Cor. 2), $BD : DA :: BDE : ADE$;

also, $CE : EA :: CDE : ADE$;

∴ (IV. 15), $BDE : ADE :: CDE : ADE$.

Therefore BDE and CDE have the same ratio to ADE; therefore (IV. 14), $BDE = CDE$.

Hence they are between the same parallels (I. 37); that is, DE is parallel to BC.

Corollary.—(See first figure.)

If $AD : DB :: AE : EC$,

by composition, $AD+DB : AD :: AE+EC : AE$,

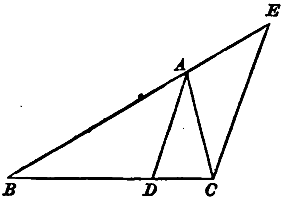
or, $AB : AD :: AC : AE$.

Proposition 3.

Theorem.—*If the vertical angle of a triangle be bisected by a straight line which also cuts the base, the segments of the base will have the same ratio, which the other sides of the triangle have to each other.*

Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD, which meets the base at D; then

$BD : DC :: BA : AC$.



Through C draw CE parallel to AD, meeting BA produced in E. Then, because AC meets the parallel lines AD, CE, the angles ACE, DAC are equal (I. 27). But the angle CAD (Hyp.) is equal to *BAD*; therefore *ACE* is equal to *BAD*. Again, because *BAE* meets the parallel lines *AD*, *EC*, the angles *BAD*, *AEC* are equal; but *BAD* was shown equal *ACE*, therefore the angles *AEC*, *ACE* are equal, and the side *AC* is equal (I. 14) to the side *AE*.

Now, because *AD* is drawn parallel to *EC*,

(V. 2), $BD : DC :: BA : AE$.

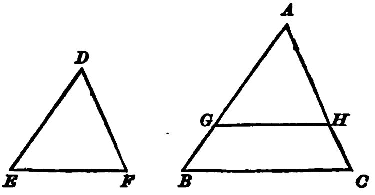
But $AE = AC$;

$\therefore$ (IV. 14), $BD : DC :: BA : AC$.

Proposition 4.

***Theorem.**—The sides about the equal angles of mutually equiangular triangles are proportional; and the homologous sides are the antecedents or consequents of the ratios.*

Let *ABC*, *DEF* be triangles equiangular to each other, having the angle *A* equal to the angle *D*, *B* equal to *E*, and *C* equal to *F*. The sides about the equal angles will be proportional.



Let the triangle *DEF* be so placed that the angle *D* will be at *A*, and the line *DE* on the line *AB*; and because the angle *D* is equal to the angle *A*, the line *DF* will fall on *AC*; *E* will be at some point in *AB*, as *G*, and *F* at some point in *AC*, as *H*; join *GH*; and because the angle *AGH* is equal to the angle *ABC*, the side *GH* is parallel (I. 26) to the side *BC*.

$\therefore$ (V. 2, Cor.), $AB : AG :: AC : AH$;

alternately (IV. 7), $AB : AC :: AG : AH$,

or, $AB : AC :: DE : DF$.

The antecedents AB and DE are opposite the equal angles C and F, and are therefore homologous, as also the consequents AC and DF.

In the same manner we may prove the sides about any other angle proportional.

***Corollary.**—Hence all mutually equiangular triangles are similar.*

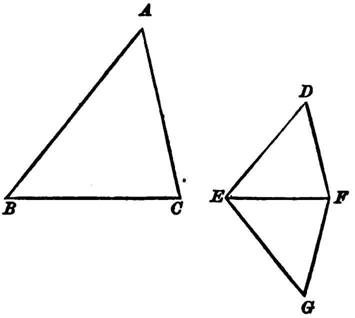
Proposition 5.

***Theorem.**—If the sides of two triangles about each of their angles be proportional, the triangles will be similar.*

Let the triangles ABC, DEF have their sides proportional,

so that $AB : DE :: BC : EF :: AC : DF$;

then will the triangles be similar.



At the points E and F in the straight line EF, make (I. 5) the angles FEG, EFG equal respectively to the angles ABC, ACB; then will the remaining angle G be equal (I. 30, Cor. 4) to the remaining angle A.

Wherefore the triangles ABC, GEF are equiangular, and (V. 4),

$$AB : BC :: GE : EF;$$

but (Hyp.) $AB : BC :: DE : EF$;

∴ (IV. 15), $GE : EF :: DE : EF$;

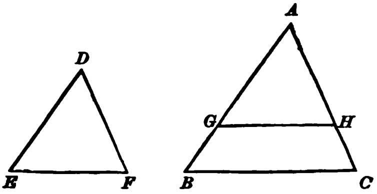
∴ (IV. 14), $GE = DE$.

Similarly, $DF = GF$.

And because the triangles DEF, GEF have one side, EF, common, and the other sides equal, each to each, the angles of one are equal (I. 4) to the angles of the other. But the angles of the triangle GEF are, by construction, equal to the angles of the triangle ABC. Therefore ABC, DEF are also equiangular, and therefore (V. 4) similar.

Proposition 6.

***Theorem.**—If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportional, the triangles will be similar.*



Let the triangles ABC, DEF have the angle A equal to the angle D, and also $AB : AC :: DE : DF$; then will the triangles ABC, DEF be similar.

Place D on A, and DE on AB; then will DF fall on AC. Let E fall at G, and F at H, and join GH. Then the triangles DEF, AGH are equal (I. 7) in all their parts.

And because $AB : AC :: DE : DF$,

∴ $AB : AC :: AG : AH$.

Alternately (IV. 7), $AB : AG :: AC : AH$.

By division (IV. 10), $BG : AG :: CH : AH$.

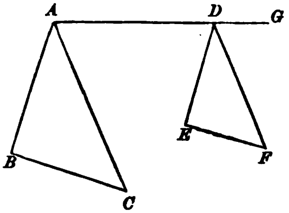
∴ (V. 2), BC is parallel to GH.

Therefore the angles of the triangle AGH are equal (I. 27) to the angles of the triangle ABC. Hence ABC, DEF are equiangular, and therefore (V. 4) similar.

Proposition 7.

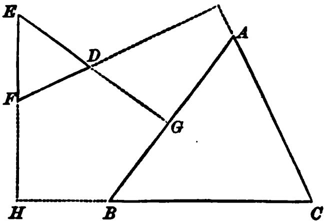
Theorem.*—*Triangles which have their sides parallel, each to each, or perpendicular, each to each, are mutually equiangular.

1. Let the triangles *ABC*, *DEF* have their sides parallel, each to each—namely, *AB* to *DE*, *AC* to *DF*, and *BC* to *EF*; they will be mutually equiangular.



Join *AD*, and produce it to *G*; then, because *GA* falls on the parallel lines *AB*, *DE*, the angles *GDE*, *GAB* are equal (I. 27). For the same reason the angles *GDF*, *GAC* are equal. Therefore (Ax. 3) the angles *BAC*, *EDF* are equal. In the same manner we may prove that the angle *B* is equal to the angle *E*, and *C* to *F*.

2. Let the sides be perpendicular to each other—namely, *AB* to *DE*, *AC* to *DF*, and *BC* to *EF*. The triangles will be equiangular.

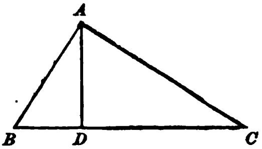


Produce *ED*, *EF* to meet *AB*, *BC* in *G* and *H*; then *EGBH* will form a quadrilateral, of which the four angles are together equal to four right angles (I. 30, Cor. 1); but *G* and *H* are (Hyp.) two right angles, therefore *GEH*, *GBH* are together equal to two right angles, and are therefore equal (I. 15) to *GBH* and *GBC*. Take away from these equals the common angle *GBH*, and *DEF* will be equal to *ABC*.

In the same manner we may prove the other angles equal.

Proposition 8.

***Theorem.**—In a right-angled triangle, if a perpendicular be drawn from the right angle to the base, the triangles on each side are similar to the whole triangle and to each other.*



Let *ABC* be a right-angled triangle, having the right angle *BAC*; and from *A* let *AD* be drawn perpendicular to the base, *BC*; the triangles *ABD*, *ADC* are similar to *ABC*, and to each other.

Because the angle *ADB* is equal to the angle *BAC*, each of them being a right angle, and the angle *ABD* is common to the two triangles, *ABD*, *ABC*, the remaining angle, *BAD*, is equal (I. 30, Cor. 4) to the remaining angle *ACB*; therefore the triangle *ABC* is equiangular to the triangle *ABD*, and therefore (V. 4) similar. In like manner it may be demonstrated that the triangles *ABC*, *ADC* are similar; and the triangles *ABD*, *ADC* being similar to *ABC*, are similar to each other.

***Corollary 1.**—The perpendicular drawn from the right angle of a right-angled triangle to the base, is a mean proportional between the segments of the base.*

For, because *ADB*, *ADC* are similar (V. 4),

$$BD : DA :: DA : DC.$$

***Corollary 2.**—Also each of the sides AB or AC is a mean proportional between the whole base and the adjacent segment.*

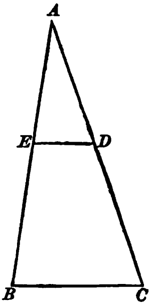
For in *ABC*, *ABD* (V. 4), $BC : BA :: BA : BD$;

and in *ABC*, *ADC* (V. 4), $BC : CA :: CA : CD$.

Proposition 9.

***Problem.**—From a given straight line to cut off any part required; that is, a part that shall be contained in it a given number of times.*

Let AB be a given straight line; it is required to cut off from AB a part that shall be contained in it a given number of times.



From A draw AC, making any angle with AB; in AC take any point D, and take AC so that it shall contain AD as many times as AB is to contain the part that is to be cut off from it; join BC, and draw DE parallel to it; then AE is the part to be cut off.

Because ED is parallel to one of the sides of the triangle ABC (V. 2),

$$CD : DA :: BE : EA.$$

By Composition (IV. 9),

$$CA : DA :: BA : EA.$$

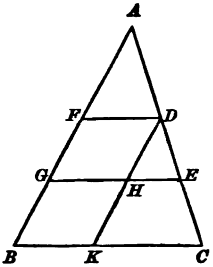
But CA is a multiple of DA; therefore BA is the same multiple of AE; and therefore whatever multiple AC is of AD, AB is of AE; wherefore from AB the part required is cut off.

Proposition 10.

***Problem.**—To divide a given straight line similarly to a given divided straight line.*

Let AB be the straight line to be divided, and AC the given divided straight line; it is required to divide AB similarly to AC.

Let AC be divided in D and E; and let AB, AC be placed so as to contain any angle, and join BC, and through D and E draw DF, EG parallel to BC; and through D draw DHK parallel to AB; wherefore each of the figures FH, HB is a parallelogram; wherefore HD is equal to GF, and HK to GB.



Because HE is parallel to KC, one of the sides of the triangle DKC,

(V. 2), $CE : ED :: KH : HD$.

But $KH = BG$, and $DH = FG$;

∴ $CE : ED :: BG : GF$.

Again, because DF is parallel to GE,

(V. 2), $ED : DA :: GF : FA$.

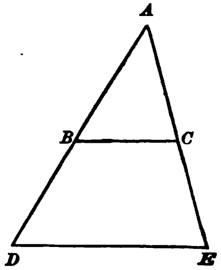
Therefore AB is divided similarly to AC.

Proposition 11.

***Problem.**—To find a third proportional to two given straight lines.*

Let AB, AC be two given straight lines; it is required to find a third proportional to AB, AC.

Place AB, AC so as to contain any angle at A, and join BC; produce AB to D, making BD equal to AC, and draw DE parallel to BC, meeting AC produced in E.



Because BC is parallel to DE,

(V. 2), $AB : BD :: AC : CE$. But $BD = AC$;

∴ $AB : AC :: AC : CE$.

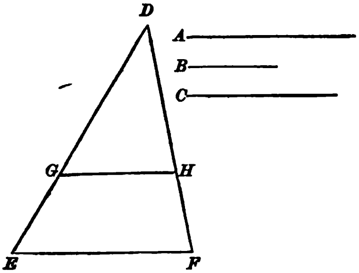
Wherefore to AB and AC a third proportional, CE, has been found.

Proposition 12.

***Problem.**—To find a fourth proportional to three given straight lines.*

Let A, B, C be three given straight lines; it is required to find a fourth proportional to A, B, C.

Take two straight lines DE, DF containing any angle EDF; and upon these make $DG = A$, $GE = B$, and $DH = C$; join GH, and draw EF parallel to it.



Then (V. 2), $DG : GE :: DH : HF$,

or, putting in the equals,

$$A : B :: C : HF.$$

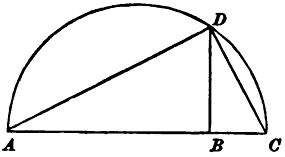
Wherefore to three lines A, B, C a fourth proportional, HF, is found.

Proposition 13.

***Problem.**—To find a mean proportional between two given straight lines.*

Let AB, BC be two given straight lines; it is required to find a mean proportional between them.

Place AB, BC in a straight line; describe upon AC the semicircle ADC, and from B draw BD at right angles to AC, and join AD, DC.

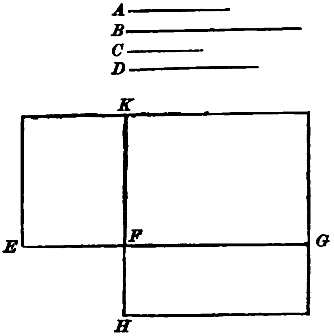


Because ADC is an angle in a semicircle, it is a right angle (III. 21). Hence DB drawn from a right angle perpendicular to the base is a mean proportional (V. 8, Cor. 1) between AB and BC.

Proposition 14.

***Theorem.**—If four straight lines be proportional, the rectangle of the extremes is equal to the rectangle of the means; and if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.*

Let A, B, C, D be four straight lines, so that $A : B :: C : D$; then the rectangle contained by A and D is equal to the rectangle contained by B and C.



Make EF equal to A, and produce it, making FG equal to B; through F draw HK at right angles to EG, making FH equal to C, and FK equal to D, and complete the parallelograms EK, KG, GH.

Now (V. 1), $EK : KG :: EF : FG$;
also (V. 1), $HG : KG :: HF : FK$.
But (Hyp.) $EF : FG :: HF : FK$.
Hence (IV. 15), $EK : KG :: HG : KG$;
$\therefore$ (IV. 14), $EK = HG$.

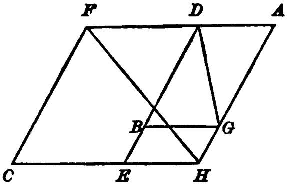
But EK is the rectangle contained by EF, FK, or by A and D, and HG is the rectangle contained by B and C. Therefore, the rectangle contained by A and D is equal to the rectangle contained by B and C.

Conversely, let EK be equal to HG then $A : B :: C : D$. For, because $EK = HG$ (IV. 14),

$EK : KG :: HG : KG$.
But (V. 1) $EK : KG :: EF : FG$,
and $HG : KG :: HF : FK$;
$\therefore$ (IV. 15), $EF : FG :: HF : FK$,
or $A : B :: C : D$.

Proposition 15.

Theorem.—*Parallelograms which are mutually equiangular are proportional to the products of their adjacent sides.*[*]



Let AB, AC be two parallelograms, which have their angles respectively equal; then

$$AC : AB :: AF \times AH : AD \times AG.$$

Place them so as to have a common angle A, and produce DB to E. Then is AE a parallelogram.

Hence (V. 1, Cor. 1), $AC : AE :: AF : AD$.

Also (V. 1, Cor. 1), $AE : AB :: AH : AG$.

Hence (IV. 13, Cor.), $\dfrac{AC}{AB} = \dfrac{AF}{AD} \times \dfrac{AH}{AG}$, or $\dfrac{AF \times AH}{AD \times AG}$,

or $AC : AB :: AF \times AH : AD \times AG$.

Corollary.—*Two triangles which have one angle of each equal, are proportional to the products of the sides about the equal angles.*

For, join FH, DG; then AFH and ADG, being halves of AC and AB, we have

$$AFH : ADG :: AF \times AH : AD \times AG.$$

Proposition 16.

Theorem.—*The area of a rectangle is equal to the product of its adjacent sides.*

Let AC be a rectangle; its area is equal to the product of AB and BC.

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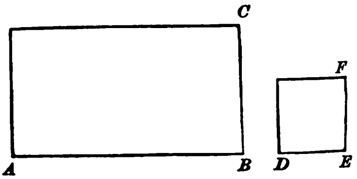
[*] By "the product of their adjacent sides," is meant the product of the number of units of the same kind in those sides.

Construct another rectangle DF;

then (V. 15), $AC : DF :: AB \times BC : DE \times EF.$

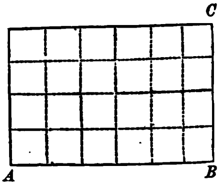


Now suppose DE and EF to be each equal to the linear unit of measure; then DF will be the square, which is the unit of measure of surfaces. Hence the above proportion becomes

$$AC : 1 :: AB \times BC : 1,$$

or

$$AC = AB \times BC.$$



Scholium.—If AB and BC be commensurable, then divide them into equal portions, each of which is a linear unit, and draw lines parallel to the sides; then it is evident that the number of square units in the surface is the product of the number of linear units in the two sides.

Corollary 1.—The area of any parallelogram is equal to the product of its base and altitude.

For it is equal to a rectangle of the same base and altitude (I. 33).

Corollary 2.—The area of a triangle is equal to one half the product of its base and altitude.

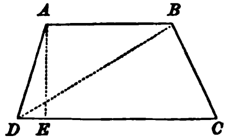
For a triangle is one half a rectangle of the same base and altitude (I. 35, Cor.).

Proposition 17.

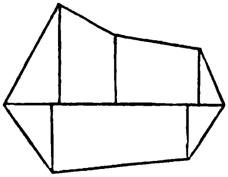
Theorem.—The area of a trapezoid is equal to the sum of the parallel sides, multiplied by one half the perpendicular distance between them.

Let AC be a trapezoid; its area is equal to the sum of AB and DC, multiplied by one half of AE.

Join DB; then the area of the triangle ADB is equal (V. 16, Cor. 2) to one half the product of AB and AE; also the triangle DBC is equal to one half the product of DC and AE. Hence the area of the whole trapezoid is equal to the sum of AB and DC multiplied by one half AE.



Scholium 1.—The area of any irregular polygon may be found by dividing it into triangles, and measuring their bases and altitudes; or by measuring a long diagonal and the perpendicular distances from the other angles, and finding the area of the triangles and trapezoids formed.

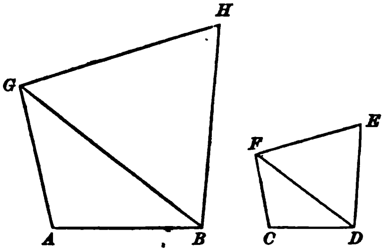


Scholium 2.—Having now shown that the rectangle contained by two sides is equal to the product of the linear units of the same kind in those sides, we can use the product of two lines instead of the rectangle contained by those lines, and *vice versâ*. Hence the (IV. 12) will now be, *If four straight lines be proportional, and also four others, the rectangles of the corresponding terms are proportional.* The truth of the (V. 14) might also have been established in a similar manner from the (IV. 1). The other proof is given to show that the algebraic and the geometrical methods lead to the same result; one by the use of abstract numbers or their equivalents, the other by considerations of magnitudes without regard to the unit which measures them. It may also be observed that AC^2 hereafter may properly mean either the square described on AC, or the second power of the number of units in AC. It has heretofore meant only the former. Hence from (IV. 12, Cor. 2) we obtain, *If four straight lines be proportional, the squares described on them will be proportional.*

Proposition 18.

***Problem.**—Upon a given straight line to describe a polygon similar to a given polygon.*

Let *AB* be a given straight line, and *CDEF* a given polygon; it is required to describe on *AB* a polygon similar to *CDEF*.



Join *DF*; and at the points *A* and *B* in the straight line *AB* make (I. 5) the angle *BAG* equal to the angle *C*, and the angle *ABG* equal to the angle *CDF*; therefore the remaining angle *AGB* is equal to the remaining angle *CFD* (I. 32, Cor. 4); wherefore the triangle *FCD* is equiangular to the triangle *GAB*. Again, at the points *G*, *B* in the straight line *GB* make (I. 5) the angles *HGB*, *HBG* equal to the angles *EFD*, *EDF*, each to each; then will the angle *H* be equal to the angle *E*, and the triangle *GBH* be equiangular to the triangle *FDE*. And because the angle *AGB* is equal to the angle *CFD*, and the angle *BGH* to the angle *EFD*, therefore the whole angle *AGH* is equal to the whole angle *CFE*. For the same reason the angle *ABH* is equal to the angle *CDE*. Therefore, the polygon *ABHG* is equiangular to the polygon *CDEF*.

The sides about the equal angles will be proportional; for because the triangles *GAB*, *FCD* are similar,

(V. 4), $BA : AG :: DC : CF$;

also $AG : GB :: CF : FD$;

also in the triangles *GBH*, *FDE*,

$GB : GH :: DF : FE$;

$\therefore$ (IV. 13), $AG : GH :: CF : FE$.

In the same manner it may be proved

$$AB : BH :: CD : DE;$$

also (V. 4), $GH : HB :: FE : ED.$

Wherefore the polygons are equiangular, and have the sides about the equal angles proportional; they are therefore similar.

In the same manner if the polygon contained five or more sides we could apply other triangles to the lines of the polygon *ABHG*, and prove the resulting polygons similar.

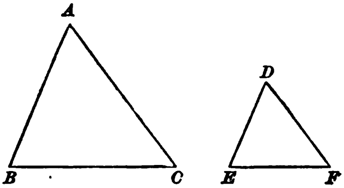
Proposition 19.

Theorem.—Similar triangles are to one another as the squares of their homologous sides.

Let *ABC*, *DEF* be similar triangles, having the angle *B* equal to the angle *E*, and let $AB : BC :: DE : EF$, so that *BC* and *EF* are homologous sides; then

$$ABC : DEF :: BC^2 : EF^2.$$

Because *ABC*, *DEF* have the angles at *B* and *E* equal to each other, we have (V. 15, Cor.)



$$ABC : DEF :: AB \times BC : DE \times EF;$$

also (Hyp.), $AB : BC :: DE : EF.$

Multiplying this, term by term, by the identical proportion

$$BC : BC :: EF : EF,$$

we have (V. 17, Sch. 2) $AB \times BC : BC^2 :: DE \times EF : EF^2;$

alternately, $AB \times BC : DE \times EF :: BC^2 : EF^2.$

But the first two terms of this proportion are proportional to *ABC* and *DEF*.

Hence (IV. 15), $ABC : DEF :: BC^2 : EF^2.$

Proposition 20.

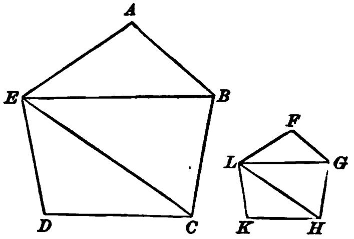
***Theorem.**—Similar polygons may be divided into the same number of similar triangles, having the same ratio to one another that the polygons have; and the polygons are to each other as the squares of their homologous sides.*

Let $ABCDE$, $FGHKL$ be similar polygons, and let AB be homologous to FG; they may be divided into the same number of similar triangles, whereof each has to each, the same ratio that the polygons have;

and $\qquad ABCDE : FGHKL :: AB^2 : FG^2$.

Join BE, EC, GL, LH; and because $ABCDE$ is similar to $FGHKL$, the angle BAE is equal to the angle GFL,

and (V., Def. 1) $\qquad BA : AE :: GF : FL$.



Wherefore (V. 6), the triangles ABE, FGL are similar, and the angle ABE is equal to the angle FGL; and because the polygons are similar, the whole angle ABC is equal to the whole angle FGH; therefore the remaining EBC is equal to the remaining angle LGH. Now, because ABE, FGL are similar (V. 4), $\qquad EB : BA :: LG : GF$.

And also because the polygons are similar,

$$AB : BC :: FG : GH;$$

$\therefore$ (IV. 13), $\qquad EB : BC :: LG : GH$;

that is, the sides about the equal angles EBC, LGH are proportional; therefore the triangle EBC is similar (V. 6) to the triangle LGH. For the same reason the triangles ECD, LHK are similar. Therefore the polygons are divided into the same number of similar triangles.

These triangles have to each other the same ratio the polygons have.

Because ABE, FGL are similar,
(V. 19), $ABE : FGL :: BE^2 : LG^2$;
also (V. 19), $EBC : LGH :: BE^2 : LG^2$;
$\therefore$ (IV. 15), $ABE : FGL :: EBC : LGH$.
Similarly, $EBC : LGH :: EDC : LKH$;
$\therefore$ also, $ABE : FGL :: EBC : LGH :: EDC : LKH$;
$\therefore$ (IV. 16), $ABE : FGL :: ABCDE : FGHKL$.

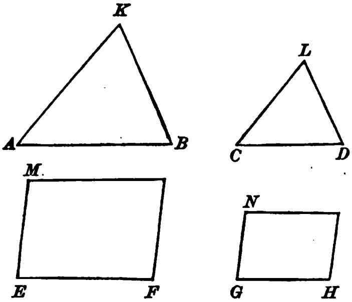
Also the polygons are to each other as the squares of AB and FG.

For $ABE : FGL :: ABCDE : FGHKL$,
and (V. 19) $ABE : FGL :: AB^2 : FG^2$;
$\therefore$ (IV. 15), $ABCDE : FGHKL :: AB^2 : FG^2$.

Proposition 21.

Theorem.—*If four straight lines be proportional, the similar figures, similarly described on each pair, will be proportional.*

Let AB, CD, EF and GH be four straight lines, so that $AB : CD :: EF : GH$; on AB and CD describe the similar figures KAB and LCD, and on EF and GH the similar figures MF and NH; then
$KAB : LCD :: MF : NH$.

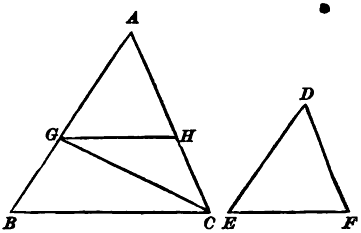


Because $AB : CD :: EF : GH$,
then (IV. 12, Cor. 2), (V. 17, Sch. 2), $AB^2 : CD^2 :: EF^2 : GH^2$.
But (V. 19) $KAB : LCD :: AB^2 : CD^2$,
and (V. 20) $MF : NH :: EF^2 : GH^2$;
$\therefore$ (IV. 15), $KAB : LCD :: MF : NH$.

Proposition 22.

***Problem.**—To find a mean proportional between two similar triangles.*

Let ABC, DEF be two similar triangles; it is required to find a mean proportional between them.



Place the triangle DEF on the triangle ABC, so that the angle D shall coincide with A; then because the triangles are similar, the side GH will be parallel to BC. Join GC.

Now (V. 1, Cor. 2), $AGH : AGC :: AH : AC$.
Similarly (V. 1, Cor. 2), $AGC : ABC :: AG : AB$.
Also (V. 2, Cor.), $AH : AC :: AG : AB$.
Hence (IV. 15), $AGH : AGC :: AGC : ABC$.

Therefore a mean proportional, AGC, has been found between the two similar triangles ABC, DEF.

Proposition 23.

***Problem.**—To cut a given straight line in extreme and mean ratio.*

Let AB be a given straight line; it is required to cut it in extreme and mean ratio.

Divide (II. 11) AB in C, so that the rectangle $AB.BC$ be equal to the square of AC.

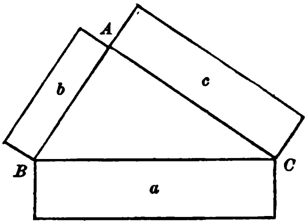
Then because $AB \cdot BC = AC^2$,

(V. 14), $AB : AC :: AC : BC$,

and AB is divided in C in extreme and mean ratio (V. Def. 2).

Proposition 24.

***Theorem.**—The figure described on the hypotenuse of a right-angled triangle is equal to the similar figures, similarly described on the other sides.*



Let ABC be a triangle right angled at A, and let a, b and c represent the similar figures on BC, AB and AC respectively; then

$$a = b + c.$$

Because c and b are similar

(V. 20), $c : b :: AC^2 : AB^2.$

By composition (IV. 9),

$$c+b : c :: AC^2+AB^2 : AC^2.$$

But (I. 42) $AC^2+AB^2=BC^2;$

$\therefore$ $c+b : c :: BC^2 : AC^2.$

But (V. 20) $a : c :: BC^2 : AC^2;$

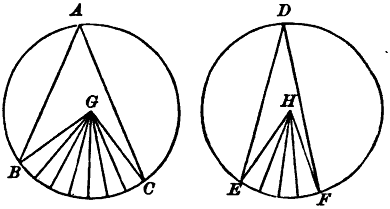
hence (IV. 15), $c+b : c :: a : c;$

$\therefore$ (IV. 14), $a=c+b.$

Proposition 25.

***Theorem.**—In equal circles, or the same circle, angles, whether at the centres or circumferences, have the same ratios which the arcs, on which they stand, have to one another; so also have their sectors.*

Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF; and the angles BAC, EDF at their circumferences; then, as the arc BC is to the arc EF, so is the angle BGC to the angle EHF, and the angle BAC to the angle EDF, and also the sector BGC to the sector EHF.



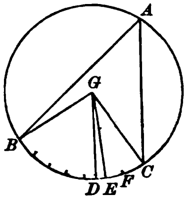
1. Let them be commensurable. Divide the arc BC into any number, as 7, equal parts, and let EF contain 4 of these parts. Through the points of division draw straight lines to the centre. Then the angles BGC, EHF will be divided into equal parts (III. 19), of which BGC will contain 7, and EHF, 4.

∴ angle BGC : angle EHF :: 7 : 4.

But arc BC : arc EF :: 7 : 4;

∴ (IV. 15), angle BGC : angle EHF :: arc BC : arc EF.

2. They are incommensurable.



Apply the smaller angle, BGD, to the larger, BGC, so that one of the sides of each shall be in the same line, BG; then, because the circles are equal, the arcs BD, BC will be in one arc; and if it be not true that angle BGC : angle BGD :: arc BC : arc BD, suppose that

angle BGC : angle BGD :: arc BC : arc BF.

Lay off the unit of measure, which we take less than DF, from B; at least one point of division, as E, will fall between D and F; join EG.

Then, according to Case I.,

Angle BGE : angle BGC :: arc BE : arc BC;

But angle BGC : angle BGD :: arc BC : arc BF;

∴ (IV. 13), angle BGE : angle BGD :: arc BE : arc BF.

But the arc *BE* is less than the arc *BF*, therefore the angle *BGE* is less than the angle *BGD*, which is impossible.

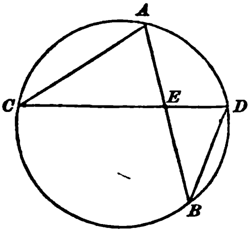
Therefore, no other arc but *BD* can be a fourth proportional to the angles *BGC*, *BGD*, and the arc *BC*.

Also because the angles (see first figures) *BGC*, *EHF* at the centres, are double (III. 16) the angles *BAC*, *EDF*, at the circumferences, therefore the angles *BAC*, *EDF*, are also proportional to the arcs *BC*, *EF*, on which they stand. So also with the sectors; for the lines drawn from the points of division will divide the sectors into equal portions, as may be proved by applying one to another. Therefore, by a similar demonstration the sectors may be proved proportional to the arcs on which they stand.

Scholium.—An angle at the centre is said to be measured by the arc which subtends it. The angles at the centre being together equal to four right angles, a right angle is measured by a quadrant. The unit used is, however, the ninetieth part of a right angle for angles, and the ninetieth part of a quadrant for arcs. This unit in both cases is called a degree; its subdivisions are minutes and seconds. Hence an angle at the circumference contains half as many degrees as the arc on which it stands, and is said to be measured by the half-arc. An angle in a semicircle therefore contains 90°; angles in the same segment (that is, which are measured by the same half-arc) contain the same number of degrees. Each side of a hexagon cuts off the sixth part of the circumference, or 60°; hence the chord of 60° is equal to radius. The opposite angles of a quadrilateral inscribed in a circle are equal to 180°, because they together stand on the whole circumference, and are measured by half of it.

Proposition 26.

* *Theorem.*—*If in a circle two chords cut each other, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.*



Let the two chords AB, CD of the circle ADB cut each other in E; then $AE \, . \, EB = CE \, . \, ED$.

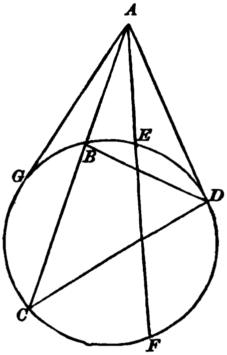
Join AC, DB; and because in the triangles AEC, DEB the angles AEC, DEB are equal to each other (I. 17), also the angles CAE, EDB, because they are in the same segment of the circle, therefore the other angles at C and B are equal (I. 30, Cor. 4), and the triangles are equiangular.

Hence (V. 4), $AE : EC :: DE : EB$;

∴ (V. 14), $AE \, . \, EB = CE \, . \, ED$.

Proposition 27.

Theorem.—If from a point without a circle, two straight lines be drawn, one of which cuts the circle and the other touches it, the rectangle contained by the whole line and the part without the circle, will be equal to the square of the line touching the circle.



From the point A draw ABC, cutting the circle CBD, and AD touching it;

then $CA \, . \, AB = AD^2$.

Join DB, DC.

Because AD touches the circle, and DB is drawn from the point of contact, the angle ADB is equal to the angle in the alternate segment DCB; also the angle A is common to the two triangles ADB, ADC; hence they are equiangular, and (V. 4),

$$AC : AD :: AD : AB;$$

∴ (V. 14), $AC \, . \, AB = AD^2$.

Corollary 1.—*If from a point without a circle, two secants be drawn to the opposite circumference, the rectangles of the whole lines and the parts without the circle, will be equal.*

Draw AEF; then $AC.AB=AF.AE$, for each is equal to the square of AD.

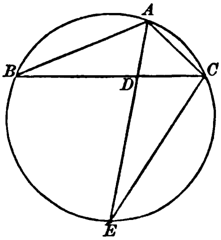
Corollary 2.—*Two tangents drawn from the same point are equal.*

Draw AG; then $AG=AD$, for the square of each is equal to $AC.AB$.

Corollary 3.—*If $AB.AC=AE.AF$, then the four points B, E, F, C are on the circumference of the same circle.*

Proposition 28.

Theorem.—*If an angle of a triangle be bisected by a straight line which also cuts the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the segments of the base, together with the square of the line bisecting the angle.*



Let ABC be a triangle, and let the angle BAC be bisected by the straight line AD; then

$$BA.AC=BD.DC+AD^2.$$

Describe (III. 29) the circle ABC about the triangle ABC, and produce AD to the circumference in E, and join EC.

Then, because the angle BAD is equal to the angle EAC, and the angle ABD to the angle AEC, they being in the same segment, the triangles ABD, AEC are equiangular.

$\therefore$ (V. 4), $BA : AD :: EA : AC$;

$\therefore$ (V. 14), $BA.AC = AD.AE.$

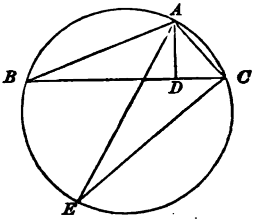
But (II. 3) $AD.AE = AD.DE + AD^2$,

and (V. 26) $AD.DE = BD.DC$;

$\therefore$ $BA.AC = BD.DC + AD^2.$

Proposition 29.

***Theorem.**—If from any angle of a triangle a perpendicular be drawn to the base, the rectangle contained by the sides of the triangle, is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*



Let ABC be a triangle, and AD the perpendicular; then

$BA.AC =$ the rectangle DA and a diameter of the circumscribed circle.

Describe the circle ABC about the triangle ABC; draw the diameter AE, and join EC.

Because the right angle BDA is equal to the angle ECA in a semicircle, and the angle ABD to the angle AEC in the same segment, the triangles ABD, AEC are equiangular.

$\therefore$ (V. 4), $BA : AD :: EA : AC$;

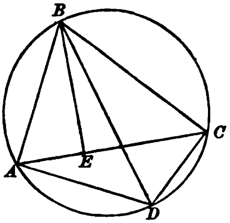
$\therefore$ (V. 14), $BA.AC = AD.AE.$

Proposition 30.

***Theorem.**—The rectangle contained by the diagonals of a quadrilateral inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

Let $ABCD$ be any quadrilateral inscribed in a circle, and let AC, DB be drawn; then

$$AC.BD = AB.CD + AD.BC.$$



Make the angle ABE equal to the angle DBC; add to each of these the angle EBD; then the angle ABD is equal to the angle EBC; and the angle BDA is equal to BCE in the same segment; therefore the triangle ABD is equiangular to the triangle BEC.

$\therefore$ (V. 4), $BC : CE :: BD : DA$;

$\therefore$ (V. 14), $BC.DA = BD.EC$.

Again, because the angle ABE is equal to the angle DBC, and the angle BAE to the angle BDC in the same segment, the triangle ABE is equiangular to the triangle DBC.

$\therefore$ (V. 4), $BA : AE :: BD : DC$;

$\therefore$ (V. 14), $BA.DC = BD.AE$.

But we have shown $BC.DA = BD.EC$;

$\therefore$ $BA.DC + BC.DA = BD.AE + BD.EC$.

But (II. 1) $BD.AE + BD.EC = BD.AC$;

$\therefore$ $BA.DC + BC.DA = BD.AC$.

EXERCISES.

1. To draw a line from the vertex of a triangle bisecting the triangle (V. 1, Cor. 2).

2. Two squares are to each other as the squares of their diagonals.

3. If a line be drawn parallel to the base of a triangle, all lines from the vertex to the base will be cut proportionally; and the segments of the base and of the parallel will be proportional to each other.

4. The perimeters of similar polygons are proportional to their homologous sides (IV. 16).

5. A line drawn parallel to the parallel sides of a trapezoid will cut the other sides proportionally.

6. If two triangles be on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal triangles and trapezoids.

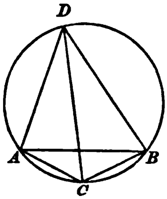
7. Let A, B and C be three points in a straight line; and through C let any straight lines pass; the perpendiculars on these lines from A and B bear a constant ratio to each other.

8. If one of the parallel sides of a trapezoid be double the other, its diagonals trisect each other.

9. Trisect a given line by use of (V. 3).

10. In the figure in (II. 11) show that four other lines besides the given one are divided in extreme and mean ratio.

11. The area of an equilateral triangle whose side is 1 is $\frac{\sqrt{3}}{4}$.



12. If the arc AB be bisected in C, and D be any point of the circumference, then $AD+DB : DC :: AB : AC$ (V. 30).

13. A circle is described about an equilateral triangle; from any point of the circumference straight lines are drawn to the angles of the triangle; then one of these lines is equal to the sum of the other two.

14. On a given line to construct a rectangle which shall be equal to a given rectangle (V. 14).

15. If from a point without the circle two lines be drawn, one cutting the circle and the other meeting, and the rectangle contained by the whole cutting-line and the part without the circle be equal to the square of the line meeting it, the latter line touches the circle (V. 27).

16. The parallelograms about the diagonal of a parallelogram are similar to the whole.

17. The base of a triangle is 6; $\frac{7}{8}$ of the triangle are cut off toward the base by a line parallel to it; what is the length of the cutting-line (V. 19)?

18. The area of a triangle is equal to its perimeter multiplied by the radius of the inscribed circle.

For other practical problems in this book see Mensuration at the end of Book VI.

BOOK VI.

REGULAR POLYGONS.—MEASUREMENT OF THE CIRCLE.

DEFINITIONS.

1. A **regular polygon** is one which is equilateral and equiangular.

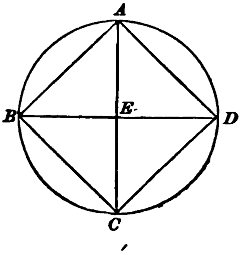
2. A polygon of five sides is called a **pentagon**; of six sides, a **hexagon**; of eight sides, an **octagon**; of ten sides, a **decagon**; of fifteen sides, a **pentedecagon**, etc., etc.

3. The **apothem** is the line drawn from the centre of a regular polygon at right angles to one of its sides.

Proposition 1.

***Problem.**—To inscribe a square in a given circle.*

Let $ABCD$ be the given circle; it is required to inscribe a square in $ABCD$.



Draw the diameters AC, BD at right angles to each other, and join AB, BC, CD, DA; because in the triangles BEA, DEA, BE, EA and the angle BEA are equal to DE, EA and the angle DEA, each to each; therefore (I. 7) BA is equal to AD; and, for the same reason, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure

ABCD is equilateral. It is also rectangular, for the straight line *BD* being a diameter of the circle *ABCD*, *BAD* is a semicircle; wherefore the angle *BAD* is a right angle (III. 21); for the same reason each of the angles *ABC*, *BCD*, *CDA* is a right angle; therefore the quadrilateral figure *ABCD* is rectangular; therefore it is a square, and it is inscribed in the circle *ABCD*.

Scholium.—Since the triangle *AED* is right-angled and isosceles, we have (I. 42, Cor. 2)

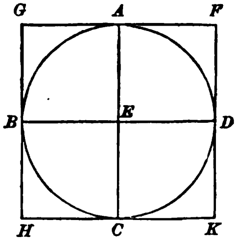
$$AD : AE :: \sqrt{2} : 1, \text{ or } AD = AE\sqrt{2}.$$

Proposition 2.

Problem.—To describe a square about a given circle.

Let *ABCD* be the given circle; it is required to describe a square about it.

Draw two diameters, *AC*, *BD*, of the circle at right angles to each other, and through the points *A*, *B*, *C*, *D* draw (III. 15) *FG*, *GH*, *HK*, *KF*, touching the circle. Hence the angles at *A*, *B*, *C*, *D* are right angles (III. 14, Cor. 2); and because the angle *AEB* is a right angle, as likewise is *EBG*, *GH* is parallel (I. 26) to *AC*; for the same reason *AC* is parallel to *FK*, and, in like manner, *GF*, *HK* may be shown to be parallel to *BED*; therefore the figures *GK*, *GC*, *AK*, *FB*, *BK* are parallelograms, and *GF* is therefore equal (I. 32) to *HK*, and *GH* to *FK*; and because *AC* is equal to *BD*, and also to each of the two *GH*, *FK*, and *BD* to each of the two *GF*, *HK*, therefore *GH*, *FK* are each of them equal to *GF* or *HK*; therefore the quadrilateral figure *FGHK* is equilateral. It is also rectangular; for *GBEA* being a parallelogram, and *AEB* a right angle, *AGB* (I. 32)

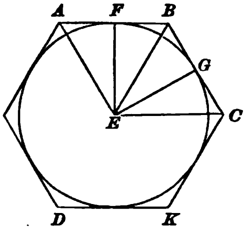


is likewise a right angle; in the same manner, it may be shown that the angles at H, K, F are right angles; therefore the quadrilateral figure $FGHK$ is rectangular, and therefore a square.

Proposition 3.

Problem.—*To inscribe a circle in a given regular polygon.*

Let $ABCD$ be a regular polygon; it is required to inscribe a circle in it.



Bisect any two adjacent angles, A and B, of the polygon by the lines AE and BE, cutting each other in E. From E draw EF, EG at right angles to AB, BC; join EC. Then, because in the two triangles FBE, GBE, EB is common, the angle EBF equal to the angle EBG, and the angles at F and G right angles, therefore EF is equal (I. 24) to EG. Also in the triangles EBA, EBC, EB, BA and the included angle EBA, are equal to EB, BC and the included angle EBC; therefore the angle EAB is equal (I. 7) to ECB. But EAB is one half an angle of the polygon, therefore ECB is also. In the same way a line from E to any angle of the polygon will bisect that angle. Therefore, in the same way that EF, EG were proved equal, all lines drawn from E perpendicular to the sides of the polygon are equal to one another, and a circle described with the centre E and radius equal to one of them, will pass through the extremities of the others, and the lines of the polygon will touch this circle (III. 14, Cor. 2). Hence the circle will be inscribed in the polygon (III. Def. 12).

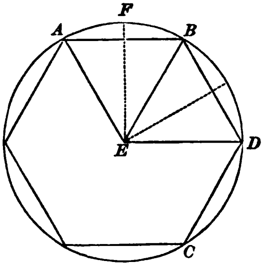
Corollary.—*From this it is manifest that the apothem is the same, no matter what side it is drawn to, and bisects the angle made by lines from the centre to the angles of the polygon.*

Proposition 4.

Problem.—*To describe a circle about a given regular polygon.*

Let *ABC* be a regular polygon; it is required to describe a circle about it.

Bisect two adjacent angles, *A* and *B*, by the lines *EA*, *BE* meeting in *E*; join *ED*; because the equal angles at *A* and *B* are bisected, therefore *EAB* is equal to *EBA*, and (I. 14) *EA* to *EB*. Also, because in the two triangles *EBA*, *EBD* the two sides *EB*, *BA* and included angle *EBA* are equal to *EB*, *BD* and included angle *EBD*, the side *EA* is equal (I. 7) to the side *ED*. In the same manner all lines drawn from *E* to the angles of the polygon are equal, and a circle from the centre *E*, at the distance of any one of them, will pass through the extremities of the others, and will be described about the polygon.



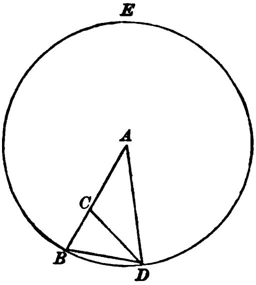
Corollary 1.—From this and the last proposition it is evident that, *The centres of the circles inscribed in a regular polygon and described about it, are in the same point.*

Corollary 2.—Also, that if the radii *EA*, *EB*, etc. be produced, they will also bisect the angles of a circumscribed polygon of the same number of sides, which will touch the circle at *F*, *G*, etc., the extremities of radii bisecting the angles at the centre. The sides of the two polygons being parallel, the angles will be equal and the polygons similar. Hence, *To draw a circumscribed polygon similar to a given inscribed polygon, bisect the angles at the centre, and from the extremities of the radii draw lines touching the circle.*

Proposition 5.

***Problem.**—To describe an isosceles triangle having each of the angles at the base double the third angle.*

Draw any circle *BDE*, and in it any radius *AB*; divide (V. 23) *AB* in *C* in extreme and mean ratio; lay down *BD* equal to *AC*, and join *AD* and *CD*; *ABD* will be the triangle required.



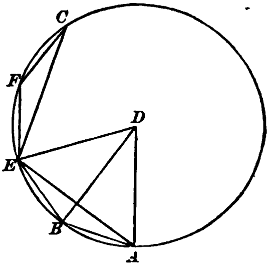
Because *AB* is divided in *C* in extreme and mean ratio, and *BD* is equal to *AC*, therefore,

$$AB : BD :: BD : BC;$$

hence the two triangles *ABD*, *DBC*, having a common angle *B*, have the sides about that angle proportional; they are therefore similar (V. 6); hence the angle *DCB* is equal to the angle *ADB*, or *ABD*, and (I. 14) *CD* is equal to *BD* or *AC*. Because *CD* is equal to *CA*, the angle *CDA* is equal to the angle *CAD*; but *BCD* is equal (I. 30) to *CAD* and *CDA*; therefore *BCD* is double *CAD*; but *BCD* is equal to *ABD* or *ADB*; hence each of the angles *ABD*, *ADB* is double *BAD*.

Proposition 6.

***Problem.**—To inscribe a regular decagon in a given circle.*



Let *ABC* be a given circle; it is required to describe a regular decagon in it.

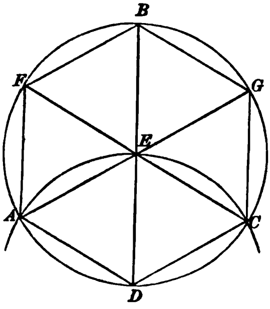
Construct (VI. 5) the isosceles triangle *DAB* on the radius *DA*, having each of the angles at the base double of the third angle. Because each of the angles *DAB*, *DBA* is double *ADB*, the two together are four times the angle *ADB*, and the three angles *DAB*, *DBA*, *ADB* are five times the angle *ADB*. Therefore the angle *ADB* is the fifth part of two right angles; that is, the tenth part of four right angles. Therefore, ten such angles as *ADB* can be exactly placed about the point *D* (I. 17, Cor. 2). But equal angles stand on equal arcs, and equal arcs are subtended by equal straight lines; therefore *AB* is one side of an equilateral decagon inscribed in the circle. And because all triangles *ADB*, *BDE* are isosceles and have equal angles at *D*, their other angles are equal. But these angles make up the angles of the decagon; therefore the decagon is also equiangular; therefore it is regular.

Corollary.—*To inscribe a regular pentagon in a given circle.*

Join the alternate angles of the decagon, and we have a regular pentagon. For all such triangles as *ABE*, *EFC* have their two sides and included angle at *B* and *F* equal in each; hence their third sides are equal, and their angles are equal. But the two angles *AEB*, *CEF*, taken from the angle of the decagon, will leave the angles of the pentagon. Therefore the pentagon is equilateral and equiangular.

Proposition 7.

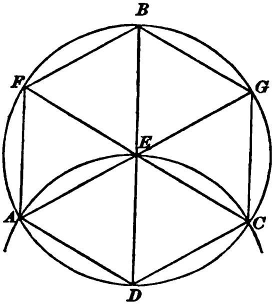
Problem.—*To inscribe a regular hexagon in a given circle.*



Let *ABCD* be a circle; it is required to describe a regular hexagon in it.

Draw any radius *ED*. From the centre *D* and radius *DE* describe a circle cutting the given circle in *A* and *C*.

Join *AE*, *CE*, and produce them, and also *DE*, to meet the circumference in *G*, *F* and *B*; join the extremities of these lines; the figure formed is a regular hexagon.



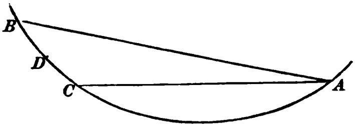
Because *DA* = *DE* = *AE*, the triangle *AED* is equiangular (I. 14, Cor.). Therefore (I. 30), the angle *AED* is one third of two right angles. For the same reason *DEC* is one third of two right angles. Therefore the whole angle *AEC* is two thirds of two right angles, and the angle *CEG* is (I. 15) one third of two right angles. Hence the vertical angles *AEF*, *FEB*, *BEG* are also each equal (I. 17) to one third of two right angles. But equal angles stand on equal arcs, and equal arcs are subtended by equal straight lines; hence, *DA*, *AF*, *FB*, *BG*, *GC*, *CD* are all equal, and the hexagon is equilateral. It is also equiangular, for the arc *DA* is equal to the arc *FB*; add to both the semicircle *BCD*, and the whole arc *BCA* is equal to the whole arc *FGD*. But equal angles stand on equal arcs; hence the angle *AFB* is equal to the angle *DAF*. In the same manner all the other angles may be proved equal, and the hexagon is equiangular. It is therefore regular.

***Corollary.**—A side of a regular hexagon is equal to the radius of the circumscribed circle.*

Hence, to construct a hexagon, describe a circle, and lay down the radius six times around it.

Proposition 8.

***Problem.**—To inscribe a regular pentedecagon in a given circle.*

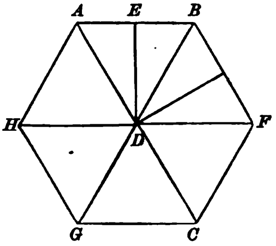


Let *AB* be one side of an equilateral triangle inscribed in a circle (III. 26), and *AC* one side of a regular pentagon (VI. 6, Cor.). If the whole circumference were to be divided into 15 equal parts, AB would subtend 5, and AC, 3. Therefore BC must contain 2; bisect (III. 23) BC in D, and lay off BD around the circle; we would have a regular pentedecagon.

Scholium.—If we bisect (III. 23) the arcs of a circle which are subtended by the sides of a square, we may form a regular polygon of 8 sides; from this, one of 16, and so on. Similarly from a decagon, we can construct polygons of 20, 40, etc., sides; from a hexagon, polygons of 12, 24, etc.; from a pentedecagon, figures of 30, 60, etc.

Proposition 9.

Theorem.—*The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



Let ABC be a polygon and DE its apothem; then the area of ABC is equal to

$$\tfrac{1}{2}(AB+BF+FC+CG+GH+HA)ED.$$

Join DA, DB, etc.; the area of the triangle DAB is equal to (V. 16, Cor. 2) $\frac{1}{2}AB\times ED$. But the apothem is the same for all sides; hence the area of the whole polygon

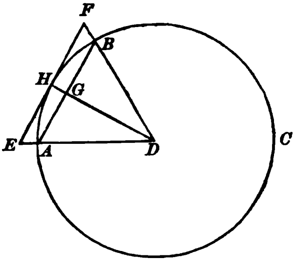
$$=\tfrac{1}{2}AB\times ED+\tfrac{1}{2}BF\times ED+\text{etc.}$$

or $$\tfrac{1}{2}(AB+BF+FC+CG+GH+HA)ED.$$

Proposition 10.

Theorem.—*About a given circle a polygon may be described, which shall differ from a similar inscribed polygon, by a space less than any given space.*

Let ABC be a circle; similar polygons may be described about ABC, and inscribed in it, whose difference shall be less than any given space.



Let AB be a side of an inscribed, and EF of a similar circumscribed polygon (VI. 4, Cor. 2); draw DH perpendicular to EF and AB; it will bisect them. The difference between the triangles DAB, DEF is the trapezoid $ABFE$, the area of which is (V. 17) $\frac{1}{2}(AB+EF)HG$. Let P represent the perimeter of the circumscribed and p of the inscribed polygon, then the difference of the polygons—that is, the sum of all trapezoids, AF, etc.—is $\frac{1}{2}(P+p)HG$, and is therefore less than $P\times HG$.

Now because $DG+GA>AD$, subtracting DG from both, $GA>AD-DG$, or $GA>HG$. By successively doubling the number of sides of the polygons, AG, and consequently GH, which will always be less than AG, may be made as small as desired, and the space $P\times HG$ may be made less than any given space; but the difference between the polygons is always less than $P\times HG$. Hence this difference may be less than any given space.

***Corollary.**—A circle may be regarded as a regular polygon of an infinite number of sides, and whatever is true of one is true of the other.*

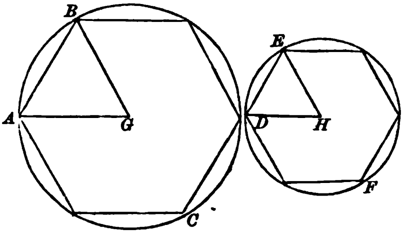
For the area of the circumscribed polygon will always be greater than the area of the circle, and of the inscribed polygon less; as we increase the number of their sides, they approach more nearly to each other, and consequently to the circle. The circles is therefore their *limit*, as the number of their sides increases indefinitely. For any finite number of sides the difference between a polygon and the circle is finite; the number of sides may increase so that the difference will become less than any given quantity; hence, as this number increases above any finite quantity—that is, to *infinity*—the difference becomes less than any finite quantity; that is, *zero*. The area of the polygon coincides with the area of the circle, the perimeter with the circumference, and the apothem with the radius. Hence, the truth of the corollary.

Proposition 11.

Theorem.—*The circumferences of circles are proportional to their radii.*

Let ABC, DEF be two circles, and AG, DH their radii; their circumferences are proportional to AG, DH.

In ABC, DEF describe the polygons ABC, DEF of the same number of sides, and draw AG, BG, DH, EH.



The angles at G and H are equal, because they are equal portions of four right angles; the polygons being regular, and having the same number of sides, the angles of one are equal to the angles of the other; hence the angles GAB, GBA are equal to the angles HDE, HED; therefore the triangles GAB, HDE are similar,

and (V. 4) alternately, $AB : DE :: AG : DH$.

If P represent the perimeter of the polygon ABC, and P' of DEF, then P and P' being equimultiples of AB and DE,

(IV. 3) $P : P' :: AG : DH$.

And this is true whatever the number of sides; it is therefore true when that number becomes infinite, in which case the perimeters of the polygons become the circumferences of the circles (VI. 10, Cor.). Hence the circumferences of circles are proportional to their radii.

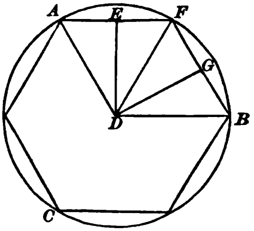
Corollary.—Hence the circumference bears a constant ratio to the diameter. This ratio is universally represented by the Greek letter π.

If C represent the circumference, D the diameter, and R the radius, then $\frac{C}{D} = \pi$, $C = \pi D$; and since $D = 2R$, $C = 2\pi R$.

Proposition 12.

Theorem.—The area of a circle is equal to one half the rectangle contained by its radius, and a straight line equal to its circumference.

Let ABC be a circle and AD its radius; the area ABC is equal to the rectangle contained by AD, and a straight line equal to one half its circumference.



Inscribe in ABC the regular polygon ABC; join DF, DB, and draw the perpendiculars DE, DG.

The area of the whole polygon is equal to one half the rectangle of DE and the perimeter of the polygon (VI. 9).

Now this being true, whatever the number of sides of the polygon, it is true if that number be infinitely increased. But in this case the polygon becomes a circle (VI. 10, Cor.), its perimeter agrees with the circumference, and its apothem with the radius. Hence the area of the circle is equal to one half the rectangle of the radius, and a straight line equal to the circumference, or the product of half the diameter and half the circumference.

Corollary 1.—The area of a circle is equal to the square of the radius multiplied by π.

Let S = area, C = circumference, D = diameter, and R = radius;

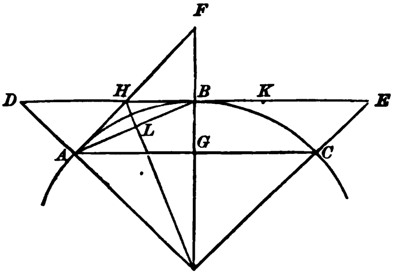
then from this Prop., $S=\frac{1}{2}R.C.$ But (VI. 11, Cor.) $C=2\pi R$. Hence, $S=\frac{1}{2}R.2\pi R=\pi R^2$.

***Corollary 2.**—Circles are proportional to the squares of their radii.*

For, let S, S' be areas of two circles, and R, R' their radii. Then (last corollary) $S:S'::\pi R^2:\pi R'^2::R^2:R'^2$.

Proposition 13.

***Problem.**—To find the approximate value of π.*



Let ABC be a circle, AC the side of a regular inscribed polygon, DE of a similar circumscribed polygon, AB, HK of regular inscribed and circumscribed polygons of double the number of sides.

Let P and p represent the perimeters of the given circumscribed and inscribed polygons, and P' and p' the perimeters of the circumscribed and inscribed polygons of double the number of sides.

Now, because $HB=HA$ (V. 27, Cor. 2), the angles HAB, HBA are equal (I. 13); also the angles HBA, BAG are equal (I. 27); therefore the angle FAG is bisected by AB;

$\therefore$ (V. 3), $AG:AF$ or $DB::BG:BF$.

But AG is contained in p as many times as DB is contained in P;

$\therefore$ (IV. 4), $p:P::GB:BF::$ (V. 2), $AH:HF$;

by composition (IV. 9), $p+P:p::AF:AH$;

hence (IV. 5), $p+P:2p::AF:2AH$ or HK.

Now, AF is contained in P as many times as HK is contained in P'.

$\therefore$ (IV. 4), $$p+P : 2p :: P : P'.$$

Hence, $$P' = \frac{2pP}{p+P}. \quad (1)$$

If now we have numerical values for p and P, we may obtain the value for P' by formula (1).

Again, in the similar triangles ABG, HBL (V. 4),

$$AG : AB :: BL : BH.$$

But AG, AB are equal parts of p and p', and BL, HB equal parts of p' and P';

$\therefore$ $$p : p' :: p' : P'.$$

Hence, $$p' = \sqrt{pP'}. \quad (2)$$

If, then, we have given a numerical value for p, and have, by formula (1), found one for P', we may find a value for p' by formula (2).

Now, we know that the perimeter of a circumscribed square in a circle whose diameter is 1, is 4, and the perimeter of an inscribed square in the same circle is $2\sqrt{2}$; hence, substituting these values for P and p in the formulæ just proved, we have

$$P' = \frac{2pP}{p+P} = \frac{16\sqrt{2}}{4+2\sqrt{2}} = 3.3137085,$$

and $$p' = \sqrt{pP'} = \sqrt{8\sqrt{2}} = 3.0614675.$$

These are perimeters of the regular circumscribed and inscribed octagons; by using them for P and p, we may obtain the values of the perimeters of polygons of 16 sides, and so on. These numbers approach each other as the number of sides increases, until for 8192 sides we have

$$P' = 3.1415928, \quad \text{and} \quad p' = 3.1415926.$$

These numbers therefore very nearly represent the length of the circumference of the circle when the diameter is 1; and since, when the diameter is 1, the circumference is π, we have, carrying out the decimal correctly to 8 places,

$$\pi = 3.14159265+.$$

This decimal has been carried out to several hundred places without coming to an end, and it may be proved by higher mathematics to be incommensurable with 1. The value usually employed in calculations is $\pi = 3.1416$. The number $\frac{22}{7}$ is a rough approximation.

EXERCISES.

1. In a given circle the inscribed square is equal to one half the circumscribed square.

2. In the figure (VI. 2) the diagonals of the square pass through the centre of the circle, and bisect the angles at *E*.

3. The regular hexagon inscribed in a circle will contain double the area of an equilateral triangle inscribed in the same circle.

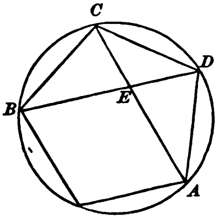
4. Show that in (VI. 5) there are two triangles which possess the required property; also that there is an isosceles triangle of which each of the angles at the base is a third part of the third angle.

5. If from any point of the circumference of a circle straight lines be drawn to the four angles of an inscribed square, the sum of the squares on these lines is double the square on the diameter (II. 9).

6. Describe a circle about a given rectangle.

7. If in (VI. 5) *DC* be produced to meet the circle in *F*, then *FB* is one side of a pentagon.

8. *ABCD* is a regular pentagon; let *AC* and *BD* meet in *E*; then *CB* is a mean proportional between *CA* and *CE*.



9. The area of a circle is to the area of the regular inscribed hexagon as π is to $\frac{3}{2}\sqrt{3}$.

10. The area of an inscribed polygon is a mean proportional between the areas of the inscribed and circumscribed polygons of one half the number of sides (V. 22).

11. The area of a circle whose radius is 1 is π.

12. The area of a sector of a circle is equal to the arc of the sector multiplied by half the radius.

13. The area of a regular inscribed hexagon is $\frac{3}{4}$ of that of a regular circumscribed hexagon.

14. A plane surface may be entirely covered by equal regular polygons of either three, four or six sides, and by no other equal regular polygons.

15. A plane surface may be entirely covered by a combination of squares and regular octagons having equal sides.

16. In a given equilateral triangle, inscribe three circles tangent to each other and to the sides of the triangle.

17. In a given circle, inscribe three equal circles tangent to each other and to the given circle.

18. If the radii of three circles be *a*, *b* and *c*, the radius of a circle containing the same area as the three together is $\sqrt{a^2+b^2+c^2}$.

19. The chord of 60° is equal to the radius.

MENSURATION OF PLANE SURFACES.

The following rules are mostly deduced from preceding propositions:

1. To find the area of a rectangle.

Multiply two adjacent sides together. (V. 16.)

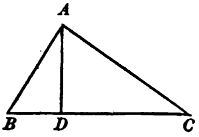
2. To find the area of a parallelogram.

Multiply one side by the perpendicular distance to the opposite side. (V. 16, Cor. 1.)

3. To find the area of a triangle.

I. *Multiply the base by one half the altitude.* (V. 16, Cor. 2.)

II. *Multiply together the half sum of the three sides, and the excess of the half sum over the three sides respectively, and take the square root of the product.**



* Let $AB=c$, $BC=a$, and $AC=b$,

then (II. 8) $a^2+c^2=b^2+2a\,.\,BD$;

hence $BD=\dfrac{a^2+c^2-b^2}{2a}$,

and $AD=\sqrt{c^2-BD^2}=\sqrt{c^2-\dfrac{(a^2+c^2-b^2)^2}{4a^2}}=\dfrac{\sqrt{4a^2c^2-(a^2+c^2-b^2)^2}}{2a}$.

Hence, $ABC=\frac{1}{2}BC\,.\,AD=\frac{1}{4}\sqrt{4a^2c^2-(a^2+c^2-b^2)^2}$. The quantity under the radical sign may be resolved into two factors, $[2ac+a^2+c^2-b^2]$ and $[2ac-(a^2+c^2-b^2)]$, and these again into

$$(a+c+b)(a+c-b) \text{ and } (b+a-c)(b+c-a).$$

Hence, the last equation becomes

$$ABC=\sqrt{\left[\frac{a+b+c}{2}\,.\,\frac{a+c-b}{2}\,.\,\frac{b+a-c}{2}\,.\,\frac{b+c-a}{2}\right]};$$

and putting $S=\dfrac{a+b+c}{2}$, $ABC=\sqrt{S(S-b)(S-c)(S-a.)}$

Hence the rule.

4. To find the area of a trapezoid.

Multiply the sum of the parallel sides by one half the perpendicular distance between them. (V. 17.)

5. To find the area of a trapezium.

Multiply a diagonal by the half sum of the perpendiculars from the opposite angles.

6. To find the area of an irregular polygon.

I. *Divide it into triangles, and find their areas separately.*

II. *Divide into trapezoids and triangles by a long diagonal and perpendiculars from the opposite angles, and find their areas separately.* (V. 17, Sch. 1.)

7. To find the area of a regular polygon.

I. *Multiply the apothem by one half the perimeter.* (VI. 9.)

II. *Multiply the square of a side by the area of a similar polygon whose side is 1.* (V. 20.)

The areas of regular polygons whose sides are unity may be calculated once for all, and tabulated.

The following table gives some of them to six places:

Triangle............	.433013	Heptagon.........	3.633912
Square..............	1.000000	Octagon...........	4.828427
Pentagon...........	1.720477	Decagon..........	7.694209
Hexagon...........	2.598076	Dodecagon.......	11.196152

8. To find the circumference of a circle.

Multiply the diameter by $\pi = 3.1416$, *or the radius by* 2π. (VI. 11, Cor.)

9. To find the diameter from the circumference.

Divide the circumference by π.

10. To find the length of an arc of a circle.

Say, As 360° *is to number of degrees in the arc, so is the whole circumference to the length of the arc.* (V. 25.)

11. To find the area of a circle.

I. *Multiply half the circumference by half the diameter.* (VI. 12.)

II. *Multiply the square of the radius by* π. (VI. 12, Cor. 1.)

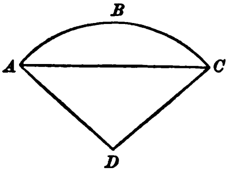
12. To find the area of a sector of a circle.

I. *Say, As four right angles is to the angle of the sector*—or, *As the whole circumference is to the arc of the sector—so is the area of the circle to the area of the sector.* (V. 25.)

II. *Multiply the arc of the sector by the radius, and divide by two.* (Proved as VI. 12.)

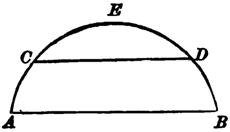
13. To find the area of a segment of a circle.

Subtract the area of the triangle ADC from the area of the sector ADCB.



14. To find the area of a zone of a circle.

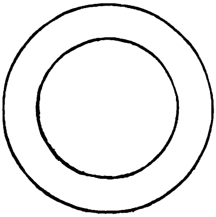
Subtract the area of the segment CED from the area of the segment AEB.



15. To find the area of a circular ring.

Subtract the area of the smaller circle from the area of the larger.

If R, R' be the radii, then the area of the ring is $\pi(R^2 - R'^2)$.



EXAMPLES.

1. The adjacent sides *AB* and *AC* of a parallelogram are 20 and 16, the distance on *AB* to the foot of the perpendicular from *C* is 5; what is the area? *Ans.* 303.8+.

2. The area of a triangle is 40 and the base is 10; what is the altitude? *Ans.* 8.

3. The sides of a rectangle are 16 and 25; what is the side of a square containing an equal area? of an equilateral triangle? *Ans.* 20; 30.39.

If a be a side of an equilateral triangle, then area is equal to $\frac{a^2}{4}\sqrt{3}$.

4. What would it cost to fence the above in the three cases at 50 cents per linear foot? *Ans.* \$41; \$40; \$45.58.

5. A side of a regular hexagon is 10; what is the area? *Ans.* 259.8.

6. The gable end of a house is 40 feet wide, the height to the eaves is 40 feet, and to the crown 50 feet; how many bricks, the face of each being 8 in. by 2 in., will be required to build it, the wall being 2 bricks thick? *Ans.* 32,400.

7. The short sides of a right-angled triangle are 25 and 36; what is the other side? what the side of an equivalent square? *Ans.* 43.8+; 21+.

8. The parallel sides of a trapezoid are 20 and 16, and the perpendicular distance between them is 6; if the other sides be produced to meet, what is the area of the whole triangle formed? *Ans.* 300.

9. In the last example the trapezoid is bisected by a line parallel to the parallel sides; how far from the longer side must it be drawn? *Ans.* 2.84+.

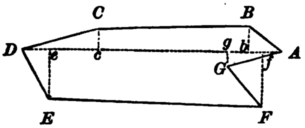
It will assist in solving this to remember that similar triangles are to each other as the squares of their homologous sides.

10. A man has three lots, each containing 10 acres; one is a square, one a circle, and one a semicircle; which requires the least fencing to enclose it? *Ans.* The circle.

11. A farmer desiring to know the contents of a quadrilateral field, steps off a diagonal, 200 steps; he then goes to the other corners, and steps perpendiculars to this diagonal, 80 and 30 steps. If each step be 3 feet, how many acres in the field? *Ans.* 2 A. 1 R. 3+r.

12. In a field AD was found to be 200 chains, Bb 20, Cc 18, Ee 40, Gg 10, Ff 42; also Af was 15, fb 10, bg 15, gc 100, ce 40, and eD 20. What is the area? *Ans.* 969 A.

It will be noted that the area is $ABb+BbcC+CcD+Eed+EefF$ $+AGg-FfgG$.



13. One side of a regular polygon is 25, and an angle is 108°; what is the area of the polygon? *Ans.* 1075.277.

14. If the diameter of the earth be 7912 miles, what is the circumference? *Ans.* 24856.28.

To obtain this answer use $\pi=3.14159$.

15. If 15° of the equator pass under the sun in one hour, how many miles will pass in 2 hours and 22 minutes? *Ans.* 2451.097 miles.

16. What is the length of one degree of the equator? *Ans.* 69.045 miles.

17. A circular plot 80 feet in diameter has a walk 8 feet wide running around it, inside the boundary; what amount of land is taken up by it? *Ans.* 1809.56.

18. A chord of a circle is 80, and its distance from the centre is 30; what is the area of the circle? *Ans.* 7854.

19. What is the side of a square equivalent to a circle 60 feet in diameter? *Ans.* 53.17.

20. A rectangle is 10 by 7; find the radius of a circle of the same perimeter. *Ans.* 5.41.

21. The hypotenuse of a right-angled triangle inscribed in a circle is 40; what is the area of the circle? *Ans.* 400π.

22. From a point without a circle a line of 8 feet is drawn touching the circle, and a line of 4 feet meeting the circle at its nearest point; what is the area of the circle? *Ans.* 36π.

23. What is the area of a sector of 25° in a circle whose radius is 44? *Ans.* 422.367.

24. What is the area of one of the segments cut off by a side of an inscribed square in a circle whose radius is 10? *Ans.* 28.54.

25. Two chords on the same side of the centre of a circle subtend angles of 60° and 40°; the radius is 14 feet; what is the area of the zone? *Ans.* 11.361 sq. ft.

26. A ladder 100 feet long is placed against a perpendicular wall; how far will the top move down by moving the bottom of it 20 feet from the wall? *Ans.* 2.03 ft.

27. A side of a square is 20 feet; find the difference between the areas of the inscribed and circumscribed circles.
Ans. Area of smaller circle.

28. Two circles, each having a radius of 10 feet, cut each other; and the angle at the centre of either, between lines drawn to the points of section, is 40°; what are the areas of the three figures formed?
Ans. 308.626; 5.534; 308.626.

29. A field is 200 metres long and 50 wide; what is its area?
Ans. 1 hectare.

30. Give the three dimensions of this book in metric units. Calculate its surface.

31. A pole 2 metres high casts a shadow 2.6 metres long; what is the height of a tree whose shadow at the same instant is 78 metres?
Ans. 60 metres.

32. A rectangle 5 centimetres long and 2 wide has a semicircle described on its two ends; what is the area of the whole figure?
Ans. 13.1416 square centimetres.

33. A horse can pasture over a circle 20 metres in radius; how much land in the circle? *Ans.* 12.5664 ares.

PART II.

SOLID GEOMETRY.

BOOK VII.

PLANES.

DEFINITIONS.

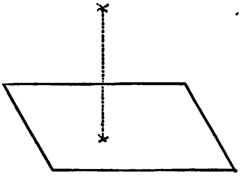
1. A **plane** or **plane surface** is one in which, if any two points be taken, the straight line joining them lies wholly in that surface.

It has no thickness, but is indefinite in extent in the other two directions.

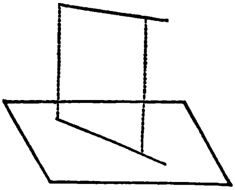
2. A straight line is **perpendicular** to a plane when it makes a right angle with every straight line which it meets in that plane.

3. A straight line is **parallel** to a plane when, being produced ever so far, it does not meet it.

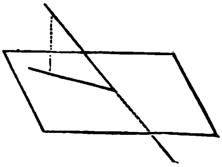
4. The **projection of a point** on a plane is the foot of the perpendicular drawn from the point to the plane.



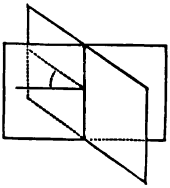
5. The **projection of a line** on a plane is the line formed by the projection of all its points.



If the line be straight, we may find the projections of two points, and join them by a straight line. The point where the line pierces the plane is one point of the projection.



6. The angle between a line and a plane is the angle between the line and its projection on the plane.



7. The angle between two planes is the angle between two lines, drawn from the same point in their common section, at right angles to it, one in each plane.

If this angle be a right angle, the planes are perpendicular to each other.

The angle between two planes is called a **diedral angle,** and it is said to be measured by the angle between the lines, drawn as described above.

8. A plane is parallel to a plane when, being produced ever so far in all directions, it does not meet it.

9. A **solid** or **polyedral angle** is the angle formed by the meeting of three or more plane angles, which are not in the same plane, at one point.

If it is made up of three angles only, it is called a **triedral angle.**

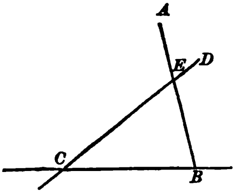
Proposition 1.

Theorem.*—One part of a straight line cannot be in a plane, and another part above it.*

If a straight line leave a plane, it must change its direction at the point where it leaves it. But a straight line is one which preserves the same direction throughout its length. Hence, if one part of a straight line be in a plane, another part cannot be above it.

Proposition 2.

Theorem.*—Any three straight lines which meet one another not in the same point, are in one plane.*



Let the three straight lines AB, CD, CB meet one another in the points B, C and E; they are in the same plane.

Let any plane pass through the straight line EB, and let the plane be turned about EB until it pass through the point C; then, because the points E and C are in the plane, the straight line EC is in it (VII. Def. 1). For the same reason BC is in it; therefore the straight lines EB, BC, CE are in the same plane; therefore the whole lines AB, BC, CD are in the same plane (VII. 1).

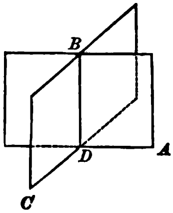
Corollary.*—Any two lines which cut each other are in the same plane. Also any three points whatever are in the same plane. Also through any three points, not in the same straight line, one, and only one, plane can be passed.*

Proposition 3.

Theorem.*—If two planes cut each other, their common section is a straight line.*

Let the two planes AB, BC cut each other, and let B and

D be two points in the line of their common section.

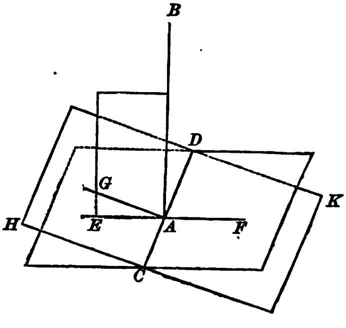


From *B* to *D* draw the straight line *BD*; and because the points *B* and *D* are in the plane *AB*, the line joining them is in that plane; for the same reason it is in the plane *BC*; it is therefore their common section.

Proposition 4.

***Theorem.**—If a straight line be at right angles to each of two straight lines at their point of intersection, it is at right angles to the plane in which these straight lines are.*

Let *AB* be at right angles to each of the straight lines *CD*, *EF*, at the point *A* of their intersection; it is at right angles to the plane in which *CD* and *EF* are.



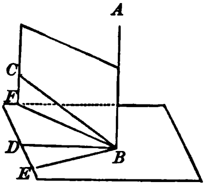
For if it be not, let some other plane passing through *CD* be at right angles to *AB*, and let it be the plane *HK*. Through *BA*, *AE* pass a plane, and let it cut the plane *HK* in the line *AG*. Then because *AB* is at right angles to the plane containing *AG*, it is at right angles to *AG* (VII. Def. 2). Therefore, *BAG* is a right angle; but *BAE* is (Hyp.) a right angle, and is equal to *BAG* in the same plane, which is impossible. Hence, the plane *HK* is not at right angles to *AB*, and in the same manner we may prove that no plane through *CD*, except such an one as passes through *EF*, is at right angles to *AB*. Therefore the plane which contains *CD*, *EF* is at right angles to *AB*.

Proposition 5.

Theorem.*—If three straight lines meet all in one point, and a straight line stand at right angles to each of them in that point, these three straight lines are in the same plane.*

Let the straight lines *AB* stand at right angles to each of the lines *BC*, *BD*, *BE* in *B*, where they meet; *BC*, *BD*, *BE* are in the same plane.

If not let *BD* and *BE* be in one plane, and *BC* be above it, and let a plane pass through *AB* and *BC*, and let it cut the plane in which *BD*, *BE* are, in the straight line *BF*; therefore the three straight lines *AB*, *BC*, *BF* are all in one plane; and because *AB* stands at right angles to *BE* and *BD* at their point of intersection, it is at right angles to the plane in which they are (VII. 4); and therefore makes a right angle with every straight line meeting it in that plane (VII., Def. 2); therefore *ABF* is a right angle; but *ABC* is (Hyp.) a right angle, and is equal to *ABF*, which is impossible; therefore the line *BC* is not above the plane in which *BD*, *BE* are; therefore *BC*, *BD*, *BE* are in the same plane.



Proposition 6.

Theorem.*—Two straight lines which are at right angles to the same plane are parallel to each other.*

Let the straight lines *AB*, *CD* be both at right angles to the plane *BDE*; *AB* is parallel to *CD*.

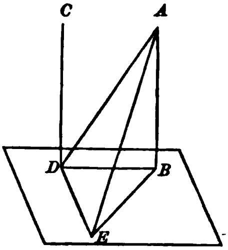
Let them meet the plane in the points *B*, *D*. Draw *DE* at right angles to *BD* in the plane *BDE*, and let *E* be any point in it; join *AE*, *AD*, *EB*. Because *ABE* is a right angle (I. 42), $AB^2+BE^2=AE^2$; and because BDE is a right angle,

(I. 42), $BE^2=BD^2+ED^2$;

∴ $AB^2+BD^2+ED^2=AE^2$.

Now (I. 42), $AB^2+BD^2=AD^2$;

∴ $AD^2+ED^2=AE^2$;

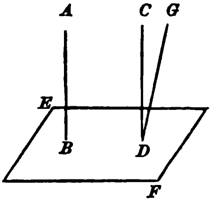


therefore ADE is (II. 8, Cor.) a right angle, and the line ED stands at right angles to the three straight lines BD, AD, CD at their point of intersection; they are therefore (VII. 5) in the same plane; but AB, BD, AD are (VII. 2) in the same plane; therefore AB, BD, DC are in the same plane, and the angles ABD, BDC are right angles; therefore CD is parallel (I. 26) to AB.

Proposition 7.

Theorem.*—If two straight lines be parallel, and one of them at right angles to a plane, the other is also at right angles to the same plane.*

Let AB and CD be two parallel straight lines, and let AB be at right angles to the plane EF; CD is also at right angles to EF.

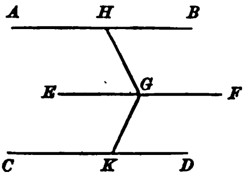


For if it be not, let DG be perpendicular to it. Then (VII. 6) DG is parallel to AB, but DC is (Hyp.) parallel to AB. DG and DC are therefore both parallel to AB, and are drawn through the same point, D, which is impossible (Ax. 11). Therefore CD is at right angles to EF.

Proposition 8.

Theorem.*—Two straight lines which are each of them parallel to the same straight line, though not in the same plane with it, are parallel to each other.*

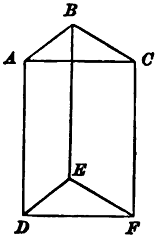
Let *AB*, *CD* be each of them parallel to *EF*, and not in the same plane with it; *AB* will be parallel to *CD*.



In *EF* take any point, *G*, from which draw, in the plane passing through *EF* and *AB*, *GH* at right angles to *EF*; and in the plane passing through *EF* and *CD* draw *GK* at right angles to *EF*. And because *EF* is perpendicular to both *GH*, *GK*, it is perpendicular to the plane *HGK* passing through them; and *EF* is parallel to *AB* and *CD*; therefore *AB* and *CD* are at right angles (VII. 7) to the plane *HGK*; therefore (VII. 6) they are parallel to each other.

Proposition 9.

***Theorem.**—If two straight lines meeting each other, be parallel to two other lines which meet each other, though not in the same plane as the first two, the first two and the other two shall contain equal angles.*



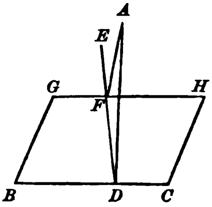
Let the straight lines *AB*, *BC*, which meet each other, be parallel to the straight lines *DE*, *EF*, which meet each other, but are not in the same plane with *AB*, *BC*; the angle *ABC* will be equal to the angle *DEF*.

Take *BA*, *BC*, *ED*, *EF*, all equal to one another, and join *AD*, *CF*, *BE*, *AC*, *DF*. Because *BA* is equal and parallel to *ED*, *AD* is equal and parallel to *BE* (I. 31). For the same reason *CF* is .equal and parallel to *BE*.

But straight lines which are parallel to the same line, though not in the same plane with it, are parallel to each other (VII. 8). Therefore *AD* is equal and parallel to *CF*, and consequently (I. 31) *AC* and *DF* are equal and parallel. Therefore the three sides of the triangle *ABC* are equal to the three sides of the triangle *DEF*; therefore the angle *ABC* is equal (I. 4) to the angle *DEF*.

Proposition 10.

Problem.*—To draw a perpendicular to a plane from a given point above it.*



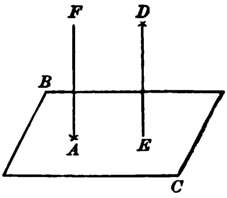
Let *A* be the given point above the plane *BH*; it is required to draw from *A* a straight line perpendicular to *BH.*

In the plane draw any straight line *BC*, and from the point *A* draw (I. 12) *AD* perpendicular to *BC.* If *AD* be also perpendicular to the plane *BH*, the thing required is done; if not, from *D* draw in the plane *BH*, *DE* at right angles to *BC*, and from *A* draw (I. 12) *AF* perpendicular to *DE*, and through *F* draw (I. 29) *GH* parallel to *BC*. *AF* is the perpendicular to the plane *BH.*

Because *BC* is at right angles to *ED* and *DA*, it is at right angles (VII. 4) to the plane *EDA*; and *GH* is parallel to *BC*; it is therefore also (VII. 7) at right angles to the plane *EDA*, and is perpendicular to every straight line which it meets in that plane; therefore *GH* is perpendicular to *AF*; but *AF* is also perpendicular to *ED*; therefore *AF* is perpendicular (VII. 4) to the plane in which *GH*, *ED* are; that is, the plane *BH*; therefore, from the point *A* the line *AF* has been drawn perpendicular to the plane *BH.*

Proposition 11.

Problem.*—To draw a line at right angles to a plane from a given point in it.*



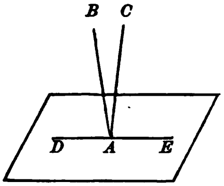
Let *A* be a given point in the plane *BC*; it is required to draw from *A* a line at right angles to *BC.*

Take any point *D* above the plane, and draw the line *DE* perpendicular (VII. 10) to the plane; and from *A* draw (I. 29) *AF* parallel to it. Then (VII. 7) *AF* is at right angles to the plane.

Proposition 12.

Theorem.*—From the same point in a plane there cannot be two straight lines at right angles to a plane upon the same side of it; and there can be but one perpendicular to a plane from a given point above it.*

For, if it be possible, let the two straight lines *AB*, *AC* be at right angles to a given plane from a point *A* in that plane; let a plane pass through *BA*, *AC*; the common section of this plane with the given plane will be a straight line passing through *A*. Let *DAE* be this line; therefore the straight lines *AB*, *AC*, *DAE* are in one plane; and because *CA* is at right angles to the plane, *CAE* is a right angle. For the same reason, *BAE* is a right angle. Therefore *CAE*, *BAE* are equal to each other, which is impossible.

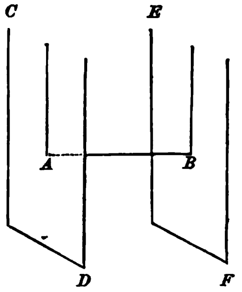


Also from the same point above a plane there cannot be drawn two perpendiculars to a plane, for if there could be two, they would be parallel (VII. 6), which is absurd.

Proposition 13.

Theorem.*—Planes to which the same straight line is perpendicular are parallel to each other.*

Let the straight line *AB* be perpendicular to each of the planes *CD*, *EF*; these planes are parallel to each other.



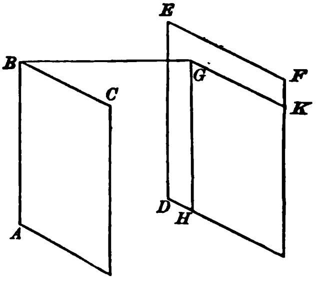
If not, they must meet when produced, and their common section must be a straight line.

Take any point in this common section and join it with *A* and *B*. Because *AB* is at right angles to the planes, it is at right angles to these lines in the plane. Therefore, there would be two angles of a triangle equal to two right angles, which is impossible. Hence, *CD* and *EF* do not meet; they are therefore parallel.

Proposition 14.

***Theorem.**—If two straight lines meeting each other be parallel to two other straight lines meeting each other, though not in the same plane with the first two, the plane which passes through the first two is parallel to the plane which passes through the other two.*

Let *AB*, *BC*, two lines meeting each other, be parallel to *DE*, *EF*, that meet each other, though not in the same plane with *AB*, *BC*. The plane through *AB*, *BC* is parallel to the plane through *DE*, *EF*.



From *B* draw (VII. 10) *BG* perpendicular to the plane through *DE*, *EF*, and let it meet that plane in *G*, and through *G* draw (I. 29) *GH* parallel to *ED*, and *GK* parallel to *EF*. And because *BG* is at right angles to the plane *DE*, *EF*, *BGH* and *BGK* are right angles (VII., Def. 2). And because *AB* is parallel to *GH* (both being parallel to *ED*) the angles *GBA*, *BGH* are equal to two right angles (I. 27); but *BGH* is a right angle, therefore *GBA* is a right angle. For the same reason *GBC* is a right angle. Hence, because *GB* stands at right angles to the two lines *AB*, *BC* at the point where they meet, it is at right angles (VII. 4) to the plane in which *AB*, *BC* are. It is also at right angles to the plane through *DE*, *EF*; but planes to which the same lines are perpendicular are parallel (VII. 13). Therefore the plane through *AB*, *BC* is parallel to the plane through *DE*, *EF*.

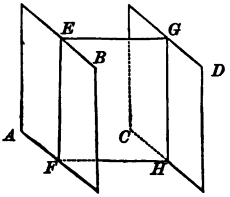
***Corollary.**—Hence, if a straight line meet two parallel planes and be perpendicular to one, it is also perpendicular to the other.*

Proposition 15.

***Theorem.**—If two parallel planes be cut by another plane, their common sections with it are parallel.*

Let the parallel planes *AB*, *CD* be cut by the plane *EFGH*, and let their common sections with it be *EF* and *GH*; *EF* is parallel to *GH*.

For the straight lines *EF*, *GH* are in the same plane, namely *EFHG*, and they do not meet, though produced, because the planes *AB*, *CD* in which they are, do not meet; therefore *EF* is parallel to *GH*.



Proposition 16.

***Theorem.**—If two straight lines be cut by parallel planes, they will be cut in the same ratio.*

Let the straight lines *AB*, *CD* be cut by the parallel planes, *GH*, *KL*, *MN*, in the points *A*, *E*, *B*; *C*, *F*, *D*;

then $$AE : EB :: CF : FD.$$

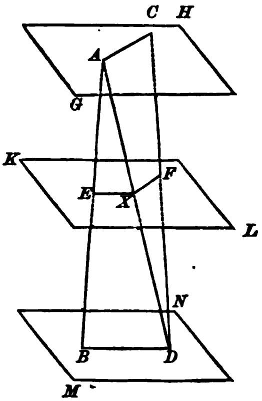
Join *AC*, *BD*, *AD*; and let *AD* meet the plane *KL* in the point *X* and join *EX*, *XF*. Because the two parallel planes *KL*, *MN* are cut by the plane *EBDX*, the common sections *BD*, *EX* are parallel (VII. 15). For the same reason, because the two parallel planes *GH*, *KL* are cut by the plane *ACD*, *AC* and *XF* are parallel; and because *EX* is parallel to *BD*, one side of the triangle *ABD*,

(V. 2), $AE : EB :: AX : XD$;

again, because *XF* is parallel to *AC*,

(V. 2), $AX : XD :: CF : FD$;

$\therefore$ (IV. 15), $AE : EB :: CF : FD$.

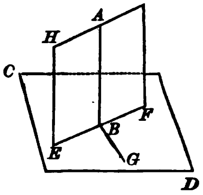


Proposition 17.

Theorem.—*If a straight line be at right angles to a plane, every plane which passes through that line is at right angles to the first-mentioned plane.*

Let the straight line *AB* stand at right angles to the plane *CD*; any plane through *AB* will also be perpendicular to *CD*.

Let *HF* be a plane passing through *AB*, and let *EF* be the common section of the two planes. In the plane *CD* draw *BG* at right angles to *EF*. Then because *AB* is at right angles to *CD*, it is at right angles to *EF* and *BG*, which it meets in that plane; but *BG* is also at right angles to *EF*. Hence *AB*, *BG* are drawn at right angles to the common section of the two planes, one in each plane, and they are at right angles to each other; therefore (VII., Def. 7) the planes *HF*, *CD* are at right angles to each other.

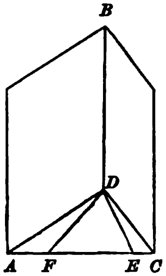


Proposition 18.

Theorem.—If two planes cutting each other be each of them perpendicular to a third plane, their common section is perpendicular to the same plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common section of the first two; BD is perpendicular to the plane ADC.

From D, in the plane ADC, draw DE perpendicular to AD and DF to DC. Because DE is perpendicular to AD, the common section of AB and ADC, and because the plane AB is at right angles to ADC, DE is at right angles to the plane AB, and therefore to DB in that plane. For the same reason, DF is at right angles to DB. Therefore DB being at right angles to DE, DF at their point of intersection is at right angles to the plane ADC (VII. 4).

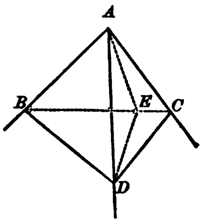


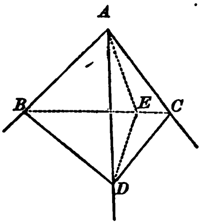
Proposition 19.

Theorem.—Any two angles of a triedral angle are together greater than the third.

Let the solid angle at A be contained by BAD, DAC, CAB; any two of these are together greater than the third.

If they are all three equal, it is evident that any two of them are greater than the third If not, let BAC be that angle which is not less than either of the other two, and is greater than one of them, DAB. Make (I. 5), in the plane, which passes through BA, AC, the angle BAE equal to the angle DAB; take any point, D, in AD, and make AE equal to AD, and through E draw BEC, cutting AB, AC



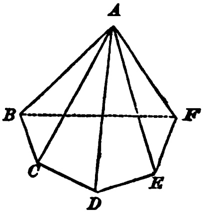


in the points *B*, *C*, and join *BD*, *CD*. And because *DA* is equal to *AE*, and *AB* is common to the two triangles *ABD*, *ABE*, and also the angle *DAB* is equal to the angle *EAB*, therefore *DB* is equal to *BE*. And because *BD*, *DC* are greater than *BC*, and one of them, *BD*, has been proved equal to *BE*, the remainder, *DC*, is greater than the remainder *EC*, therefore the angle *DAC* is greater (I. 23) than the angle *EAC*, and by construction *DAB* is equal *BAE*; therefore the angles *DAB*, *DAC* are together greater than *BAE*, *EAC*—that is, than *BAC*; but *BAC* is not less than either of the others; therefore *BAC* with either of them is greater than the other.

Proposition 20.

Theorem.*—The plane angles which contain any solid angle are together less than four right angles.*

Let *A* be a solid angle contained by any number of plane angles *BAC*, *CAD*, *DAE*, *EAF*, *FAB*; these together are less than four right angles.



Let the planes which contain the solid angle *A* be cut by another plane, and let the section of them by this plane be the rectilineal figure *BCDEF*.

Let $M =$ the sum of all the angles at *A*;
$n =$ the number of angles at *A*;
$S =$ the sum of the angles at the bases of the triangles;
$P =$ the sum of the angles of *BCDEF*;
$R =$ one right angle.

Because the angle at B is triedral, the two angles ABC, ABF are together greater (VIII. 10) than CBF. In the same way all the angles at the bases of the triangles having a common vertex at A, are greater than the angles of $BCDEF$, or

$$S>P.$$

Now (I. 30), the angles of ABC are equal to two right angles, and the same is true of the other triangles; hence,

$$M+S=n\times 2R;$$

also (I. 30, Cor. 1), $P=n\times 2R-4R$,

or, $P+4R=n\times 2R$;

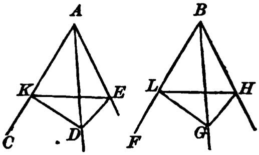
hence, $M+S=P+4R$.

But $S>P$; $\therefore\ M<4R$.

Proposition 21.

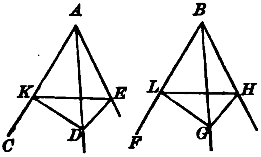
***Theorem.**—If the plane angles which constitute one triedral angle be equal, each to each, to the plane angles which constitute another, the faces of the two angles are equally inclined to each other.*

Let A and B be two triedral angles, of which the plane angles CAD, FBG, as also DAE, GBH, and CAE, FBH are equal to each other; then the planes containing the equal angles are equally inclined to each other.



In AC take any point K, and draw, in the planes CAD, CAE, KD, KE at right angles to AC; then DKE will be (Def. 7) the inclination of those planes; make BL equal to AK, and draw GL, LH in the planes FBG, FBH, at right angles to BF; then GLH will be the inclination of those planes.

Because in the triangles AKD, BLG, AK is equal to BL, the angle KAD to the angle LBG (Hyp.), and the angles AKD, BLG, right angles, therefore (I. 8) KD is equal to LG, and AD to BG. For the same reason KE is equal to LH, and AE to BH. Then, in the triangles ADE, BGH, the two sides DA, AE are equal to GB, BH, and their included angles are equal; therefore (I. 7), DE is equal to GH. Lastly, in the triangles DKE, GLH, because the three sides of one are equal to the three sides of the other, the angles DKE, GLH are equal (I. 4) to each other; but these angles are the inclination of the planes DAC, CAE, and the planes GBF, FBH, to each other; therefore the planes are equally inclined. And in the same way we may prove it of the other faces.



EXERCISES.

1. To pass a plane perpendicular to two parallel lines.

2. If two planes, which intersect, contain two lines which are parallel to each other, one in each plane, the common section of the planes will be parallel to the lines.

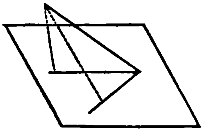
Pass a plane perpendicular to the parallel lines.

3. To pass a plane through a given line perpendicular to a given plane.

4. If a line be parallel to one plane and perpendicular to another, the planes are at right angles to each other.

5. The perpendicular is the shortest line drawn from a point to a plane. Oblique lines drawn at equal distances from the perpendicular are equal; and of two oblique lines unequally distant from the perpendicular, the more remote is the greater.

6. The acute angle which a line makes with its own projection on a plane, is the least angle which it makes with any line on the plane.



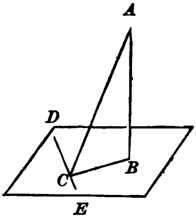
See Def. of projection, and use the last Ex. and (I. 23).

7. If a plane be passed through a diagonal of a parallelogram, the perpendiculars to this plane from the extremities of the other diagonal are equal.

Pass a plane through the perpendiculars and the other diagonal, and work in equal triangles.

8. If each of the projections of a line on two intersecting planes be straight, the line itself is a straight line.

9. If, from a point *A*, a perpendicular be drawn to a plane meeting it at *B*, and from *B* a perpendicular be drawn on a straight line *DE* in the plane, meeting it in *C*, then *AC* is perpendicular to *DE*.



10. Within a given triangle is inscribed another triangle; then the sum of the angles subtended by the sides of the interior triangle at a point not in the plane of the triangle, is less than the sum of the angles subtended at the same point by the sides of the exterior triangles (VII. 19).

11. A straight line is drawn within a triedral angle; the angles which this line makes with the lines which form the sides of the angle, are greater than the half sum of the three angles forming the triedral angle (VII. 19).

BOOK VIII.

POLYEDRONS, CYLINDERS, CONES.

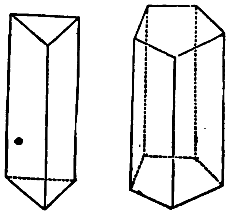
DEFINITIONS.

1. A **solid** is a figure having length, breadth and thickness.

2. A **polyedron** is a solid bounded by plane surfaces.

3. A **prism** is a polyedron of which two opposite faces, called bases, are equal, similar, and parallel polygons, and the other faces are parallelograms.

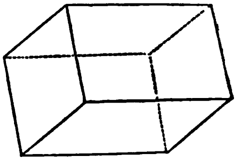
If the lateral edges stand at right angles to the base, it is said to be a **right** prism. If the base be a triangle, it is called a **triangular** prism; if a pentagon, a **pentangular** prism, etc., etc.



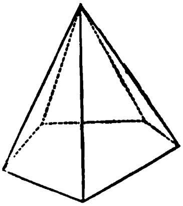
4. A **parallelopiped** is a prism whose base is a parallelogram.

Corollary.—Hence, *all its faces are parallelograms.*

If all its faces be rectangles, it is said to be **rectangular.**

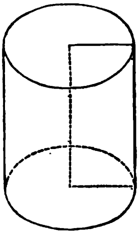


If all its faces be squares, the solid is called a **cube.**



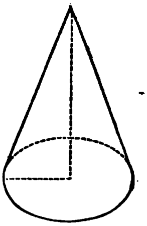
5. A **pyramid** is a polyedron, of which one of its faces, called the **base**, is any polygon, and the other faces are triangles having a common **vertex**, which is not in the plane of the base. If the base be a regular polygon, and the line from the vertex to the middle of the base be at right angles to its plane, the pyramid is a **right** pyramid.

If the base be a triangle, it is called a **triangular** pyramid, etc. etc.



6. A **cylinder** is a solid formed by the revolution of a rectangle about one of its sides.

7. This side is the **axis** of the cylinder; the circular ends are the **bases.**



8. A **cone** is a solid formed by the revolution of a right-angled triangle about one of the sides adjacent the right angle.

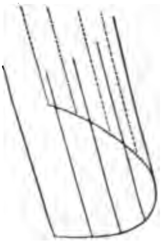
9. This side is the **axis** of the cone; the **base** is the circle formed by the other side, and the other extremity of the axis is the **vertex.**

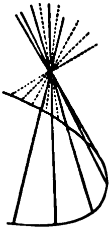
10. The **slant height** of a cone is the distance from any point of the circumference of the base to the vertex. The **slant height** of a right pyramid is the distance of any side of the perimeter of the base from the vertex.

11. The **frustum** of a cone or pyramid is the portion cut off toward the base by a plane parallel to the base.

12. **Similar solids** are such as are made up of the same number of similar planes, similarly placed. **Similar cylin-**

ders and **cones** are such as have their altitudes and diameters of their bases proportional to each other.





13. The terms **cylinder** and **cone** have often a wider meaning than that just given. If a straight line be moved on *any* curve, so as to be always parallel to its original position, the surface generated is a **cylindrical** surface. If moved so that all the different positions of the line pass through a point not in its plane, the surface is a **conical** surface. The different positions of the generating line are called **elements** of the surface. The elements of a cone may be continued beyond the vertex when another cone is formed similar to the first. If the base be a circle, and the elements of the cylinder be perpendicular to the base, the ordinary cylinder is formed, which is therefore said to be a right cylinder with a circular base. If the line from the vertex of the cone to the centre of the circular base be perpendicular to the plane of the base, the cone is a right cone.

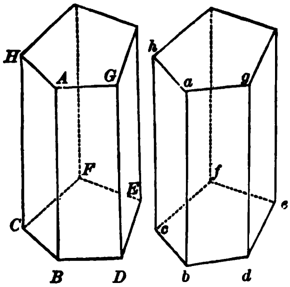
14. A prism is **inscribed** in a cylinder when all the lateral edges of the prism are elements of the cylinder. A pyramid is **inscribed** in a cone when all the lateral edges of the pyramid are elements of the cone.

Proposition 1.

Theorem.—If two prisms have the three faces which bound any solid angle in each, equal, similar and similarly placed, the prisms will be equal.

Let the two prisms *A-BDEFC, a-bdefc* have the three faces which bound the solid angles *B, b,* equal, similar and similarly placed. The prisms will be equal to each other.

Apply the prism *ACE* to the prism *ace*, so that *B* shall be on *b*, and the bases coincident. And because the three angles which contain the angle *B* are respectively equal to the three which contain the angle *b*, their faces are equally inclined to each other (VII. 21). Therefore the faces *AD*, *AC* will coincide with *ad*, *ac*; and because they are equal and similar, the lines *AG*, *AH* will coincide with *ag*, *ah*. Therefore the upper bases, being equal and similar to the lower bases and to each other, will coincide altogether, and hence the other faces will coincide, and the prisms will coincide altogether, and are therefore equal to each other.

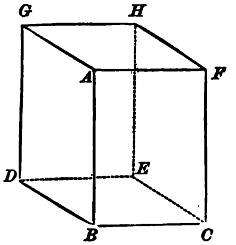


Proposition 2.

***Theorem.**—The opposite faces of a parallelopiped are equal and similar figures.*

Let *A-BCED* be a parallelopiped; its opposite faces are equal and similar.

The two bases *GF*, *DC* are equal and similar by (Def. 3). Because *AC* is a parallelogram, *AB* is equal and parallel to *FC*; for the same reason, the other sides of the face *AD* are equal and parallel to the other sides of the face *FE*. Therefore the angles of one face are equal to those of the other (VII. 9). Therefore the faces are equal



and similar; and in the same way it may be proved that *AC* and *GE* are equal and similar.

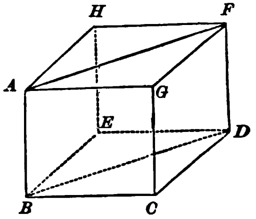
Scholium.—Hence any two opposite faces of a parallelopiped may be considered as bases.

Proposition 3.

Theorem.—*If a plane be passed through the diagonals of two of the opposite faces of a parallelopiped, it will divide the parallelopiped into two equal triangular prisms.*

Let *A-BCDE* be a parallelopiped, and let a plane be passed through *AF* and *BD*, the diagonals of the opposite faces. The two prisms formed will be equal to each other.

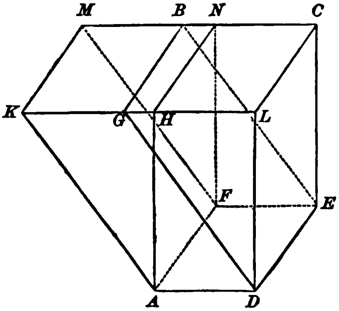
Because *EC* is a parallelogram and *BD* is its diagonal, the triangles *BED*, *BCD* are equal and similar; but *BED* is equal and similar to *AHF*, being halves of the equal and similar parallelograms *EC*, *HG*. Hence *BCD*, *AHF* are also equal and similar. Also, the face *AC* is equal and similar (VIII. 2) to the face *EF*, and *AE* to *GD*. Therefore the three faces which bound the solid angle *C* are equal and similar to the three faces which bound the solid angle *H*. They are also similarly placed; therefore the prism *A-BCD* is equal to the prism *A-BED* (VIII. 1).



Proposition 4.

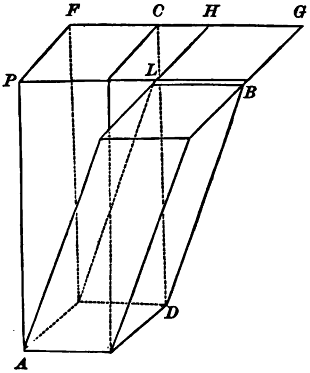
Theorem.—*Parallelopipeds which are upon the same base and of the same altitude are equal to one another.*

Case I.—Let *AB*, *AC* be two parallelopipeds upon the same base, *ADEF*, and of the same altitude; and let their upper bases, *MG*, *CH*, lie between the same parallels *MC*, *KL*. *AB* and *AC* are equal to each other. Because *KG* and *HL* are both equal to *AD*, they are equal to each other; hence *KH* is equal to *GL*; and because they are opposite sides of a parallelogram, *AH* is equal to *DL*, and *AK* to *DG*. Therefore the triangles *AHK*, *DLG* are equal and similar; also the face *AM* is equal and similar (VIII. 2) to the face *DB*, and *AN* to *DC*. Therefore the two triangular prisms, *AKH-FMN*, *DGL-EBC*, have the three faces which bound the solid angles at *A* and *D* respectively equal to each other; therefore the prisms are equal (VIII. 1).



From the polyedron *KADL-C* take each of these equal prisms; there will remain the parallelopipeds *AB* and *AC*, which are therefore equal to each other.

Case II.—If the upper bases do not lie between the same parallel lines, produce their sides to intersect, so as to form the parallelogram *LG*. The parallelograms *LG*, *AD* will be equal to each other, because their sides are respectively equal and parallel. A parallelopiped which would have *AD* for its lower base and *LG* for its upper would be equal (Case I.) to each of the parallelopipeds *AB*, *AC*. They are therefore equal to each other.

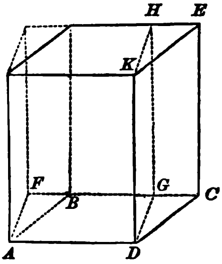


Proposition 5.

Theorem.—Any parallelopiped is equal to a rectangular parallelopiped of equal base and altitude.

Let *ABCD* be the base* of any parallelopiped. The parallelopiped is equal to a rectangular one on an equal base and of the same altitude.

On the base *ABCD* construct the parallelopiped *AE*, of the same altitude as the given one, and having each of the faces perpendicular to the plane of the base. This will be equal (VIII. 4) to the given parallelopiped. In the plane of the base draw *AF*, *DG* at right angles to *AD*, meeting *BC* in *F* and *G*. On *AG* construct a rectangular parallelopiped, *AH*, of the same altitude as the others.



The rectangle *AG* is equal (I. 33) to the parallelogram *AC*; also the parallelopipeds *AH*, *AE* are equal to each other, because they are on the same base, *AK*, and of the same altitude; but *AE* is equal to the given parallelopiped; therefore *AH* is equal to the given parallelopiped, and it is on the base *AG* equal to *AC*, and has the same altitude.

Corollary.—Parallelopipeds on equal bases and of the same altitude are equal to each other.

For they are all equal to the same rectangular one.

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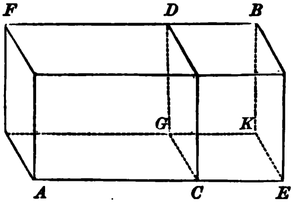
* The parallelopiped is not constructed, because the number of lines would cause confusion.

Proposition 6.

***Theorem.**—If a rectangular parallelopiped be cut by a plane parallel to its opposite faces, the solids formed are proportional to their bases.*

Let the rectangular parallelopiped, AB, be cut by a plane, CD, parallel to EB and AF;



then $AD : CB :: AG : CH.$

CASE I.—Let the lines AC and CE be commensurable, and let AC contain the common unit any number of times, as 7, and CE any number of times, as 4; through the points of division conceive planes to be passed parallel to CD. Then AD will be divided into 7 equal prisms (I. 34), (VIII. 5, Cor.), and CB into 4.

Therefore, $AD : CB :: 7 : 4.$

But $AC : CE :: 7 : 4;$

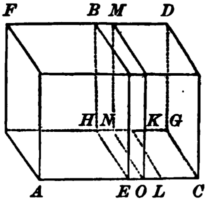
∴ (IV. 15), $AD : CB :: AC : CE.$ (1)

But (V. 1, Cor.) $AC : CE :: AG : CH;$

∴ (IV. 15), $AD : CB :: AG : CH.$

CASE II.—Let the lines AC and CE be incommensurable. Suppose the prism CB be applied to the prism AD, so that CD shall coincide with AF.

(Up to this point refer to last figure.)



If it be not true that

$AD : AB :: AG : AH,$

suppose that $AD : AB :: AG : AK.$

Beginning at A, divide AC into equal portions by some unit smaller than EL. At least one point of division, as O, will fall between E and L. Through O let a plane be passed parallel to EB, and let AM be the prism formed.

Then (Case I.) $AM : AD :: AN : AG$.
But by supposition, $AD : AB :: AG : AK$;
∴ (IV. 13), $AM : AB :: AN : AK$.

But AM is greater than AB, therefore AN is greater than AK. But AN is less than AK, which is impossible.

Therefore it is not true that

$$AD : AB :: AG : AK.$$

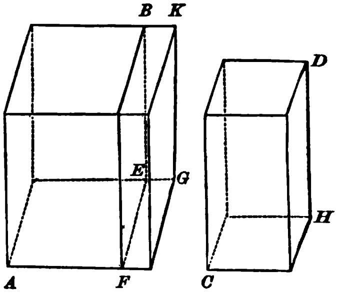
Hence $AD : AB :: AG : AH$.

Corollary.—From Proportion (1) of Case I. it is evident that, *Rectangular parallelopipeds on the same base are proportional to their altitudes.*

Proposition 7.

Theorem.—*Rectangular parallelopipeds of the same altitude are proportional to their bases.*

Let AB, CD be two rectangular parallelopipeds of the same altitude;
then $AB : CD : AE : CH$.

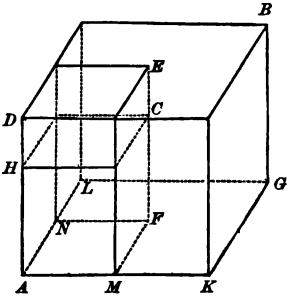


On EF construct the rectangle FG equal to CH, and upon FG construct the rectangular parallelopiped FK of the altitude of the prisms. Then (VIII. 6), $AB : FK :: AE : FG$.
But (VIII. 5, Cor.),
$FK = CD$, also $FG = CH$;
∴ $AB : CD :: AE : CH$.

Proposition 8.

Theorem.—*Any two rectangular parallelopipeds are proportional to the products of their bases and altitudes.*

Let AB, AC be two rectangular parallelopipeds. Place them so as to have a common angle A, and produce, if necessary, the sides of AC so as to cut the plane DB in the parallelogram DE. Then AE will be another parallelopiped having the same altitude as AB, and the same base as AC.



Now (VIII. 6, Cor.),

$$AE : AC :: AD : AH;$$

also (VIII. 7),

$$AB : AE :: AG : AF;$$

∴ (IV. 13, Cor.),

$$AB : AC :: AD \times AG : AH \times AF.$$

Corollary 1.*—Rectangular parallelopipeds are proportional to the products of their three dimensions.*

For (V. 16) the bases are equal to the products of their adjacent sides; hence the last proportion becomes

$$AB : AC :: AD \times AK \times AL : AH \times AM \times AN.$$

Corollary 2.*—The volume of any rectangular parallelopiped is equal to the product of its three dimensions, or of its base and altitude.**

For suppose each of the three dimensions of AC to be the linear unit of measure, then the volume AC will be the cubic unit, and the last proportion becomes

$$AB : 1 :: AD \times AK \times AL : 1 \times 1 \times 1,$$

or

$$AB = AD \times AK \times AL.$$

Corollary 3.*—The volume of any parallelopiped is equal to the product of its base and altitude.**

For any parallelopiped is equal to a rectangular one of equal base and altitude.

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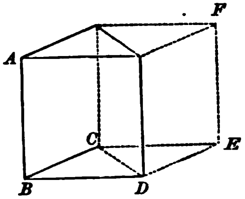
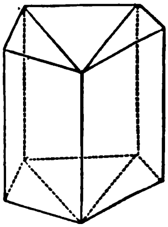
* By this is meant the product of the square units of the base and the linear units of the altitude.

Proposition 9.

***Theorem.**—The volume of any prism is equal to the product of the area of its base and its altitude.*



Let *A-BCD* be a prism, and first let it be triangular. Complete the parallelopiped *BF*. It is equal to the area of its base multiplied by its altitude (VIII. 8, Cor. 3). The prism *A-BCD* is one half (VIII. 3) the parallelopiped *BF*, and its base *BCD* is one half the base *BE*. Therefore the volume of *A-BCD* is equal to the product of the area of its base and its altitude.



Next let it have any number of sides. It may be divided by planes into triangular pyramids having a common altitude. As each of these is equal to the area of its base multiplied by its altitude, their sum is equal to the sum of their bases, or the whole base, multiplied by the common altitude.

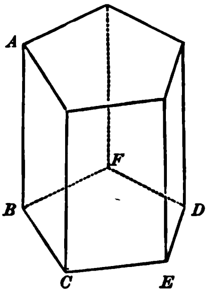
***Corollary.**—Prisms on equal bases and of the same altitude are equal to one another.*

Proposition 10.

***Theorem.**—The lateral surface of a right prism is equal to the perimeter of its base multiplied by its altitude.*

Let *A-BCD* be a right prism; its lateral surface—that is, the sum of all the parallelograms forming its sides—is equal to the perimeter multiplied by its altitude.

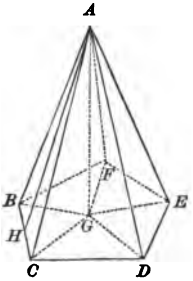
Because the prism is a right prism (Def. 3) the angle *ABC* is a right angle. Hence the figure *AC* is a rectangle. Similarly, all the lateral faces are rectangles, and are equal to the product of their length and breadth (V. 16). But they have a common altitude *AB*, and their other sides *BC*, *CE*, etc. make up the perimeter of the base of the prism. Hence the sum of the areas of all of these rectangles is equal to the perimeter of the base of the prism multiplied by its altitude; or the lateral surface of the prism $= AB(BC + CE + ED + DF + FB)$.



Proposition 11.

***Theorem.**—The lateral surface of a right pyramid is equal to the perimeter of its base multiplied by one half its slant height.*

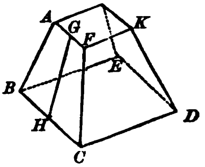
Let *A-BCE* be a right pyramid. Its lateral surface—that is, the sum of the areas of all the triangles forming the sides—is equal to the perimeter of its base multiplied by one half the slant height.



Because this is a right pyramid (Def. 5), *BCDEF* is a regular polygon, and the perpendicular *AG* passes through the middle of the base. Hence, *GB* is equal to *GC*; therefore, *AB* is equal (I. 42) to *AC*. Similarly, the other edges are equal, and the triangles forming the lateral surface are all equal (I. 4) to one another.

Now, because the slant height *AH* is (Def. 10) the distance of *BC* from *A*, it is at right angles to *BC*. It is therefore the altitude of the triangle *ABC*. The area of this triangle is therefore equal to one half the product of *BC* and *AH*. Hence the sum of the triangles is equal to the sum of their bases, or perimeter of the base of the pyramid, multiplied by one half its slant height, or the common altitude AH; or the lateral surface of the pyramid

$$=\frac{1}{2}AH(BC+CD+DE+EF+FB).$$

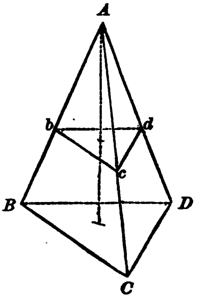


Scholium.—Let A-$BCDE$ be the frustum of a pyramid, and GH its slant height. Then, because BD and AK are parallel planes and the lateral faces meet them, their intersections are parallel. Hence, each of the faces BF, CK, etc., is a trapezoid. Now, the area of a trapezoid is equal to the sum of the parallel sides multiplied by half the distance between them (V. 17). But the distance between them is the slant height of the frustum. Hence, the sum of all the trapezoids, or *the lateral surface of the frustum, is equal to the sum of the perimeters of the two ends multiplied by half the slant height.*

Proposition 12.

Theorem.—*If a plane be passed cutting a pyramid parallel to the base—*

1. *The edges and the altitude will be cut proportionally.*
2. *The section is a figure similar to the base.*



1. Let a plane cut the cone A-BCD parallel to the base, making the figure bcd; the edges and altitude will be cut proportionally.

For, suppose a plane to pass through A parallel to the plane of the base; then the edges and the altitude, being cut by three parallel planes, will be cut proportionally (VII. 16).

2. BCD will be similar to bcd. For, because BC is parallel (VII. 15) to bc, the triangles ABC, Abc are similar.

∴ (V. 4) and alternately, $AB : Ab :: BC : bc$.

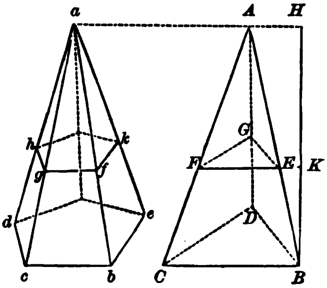
For the same reasons, $AB : Ab :: BD : bd$;

∴ (IV. 15) and alternately, $BC : BD :: bc : bd$.

Also the angle *CBD* is equal (VII. 9) to the angle *cbd*, and the sides about the equal angles are proportional; and the same is true of any other angles of the figures. They are therefore similar.

Proposition 13.

***Theorem.**—If two pyramids, having equal bases and altitudes, and their bases in the same plane, be cut by a plane parallel to the base, the figures formed will be equal to each other.*



Let pyramids *A-BCD*, *a-bcde*, which have equal bases *BCD*, *bcde*, and the same altitude *BH*, and their bases in the same plane, be cut by a plane parallel to the base at any distance *BK* from it; the sections *EFG*, *fghk* will be equal.

Because *BCD*, *EFG* are (VIII. 12) similar figures (V. 19),

$$BCD : EFG :: BC^2\ EF^2;$$

also (V. 4), (V. 17, Sch.),

$$BC^2 : EF^2 :: AB^2 : AE^2 :: \text{(VII. 16)}, HB^2 : HK^2;$$

∴ (IV. 15), $BCD : EFG :: HB^2 : HK^2$.

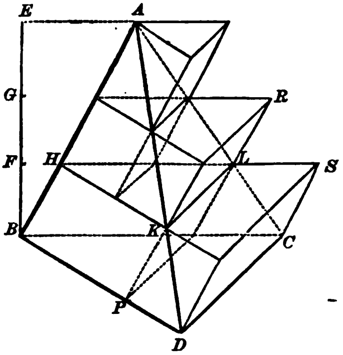
Similarly, $bcde : fghk :: HB^2 : HK^2$;

∴ (IV. 15), $BCD : EFG :: bcde : fghk$.

But (Hyp.) *BCD* is equal to *bcde*; therefore *EFG* is equal to *fghk*.

Proposition 14.

Theorem.—*A series of prisms may be inscribed in a pyramid, and another series described about it, which shall differ from each other by a solid less than any assignable solid.*



Let A-BCD be a pyramid, and BE its altitude. Divide BE into any number of equal portions, and through the points of division pass planes parallel to the base of the pyramid. On these sections as lower bases describe a series of prisms; these may be said to circumscribe the pyramid. On these sections as upper bases describe a series of prisms; these may be said to be inscribed in the pyramid.

Now, because the inscribed and circumscribed prisms HKL-P and HKL-R on any base HKL have the same altitude, they are equal (VIII. 4) to each other. Hence, all the circumscribed prisms have equal inscribed prisms, except the prism on the base BDC. Hence, the sum of the circumscribed prisms exceeds the sum of the inscribed prisms by the prism BDC-S.

Now, by making the divisions on BE small enough, the prism BDC-S may be less than any assignable solid. Hence, the difference between the inscribed and circumscribed prisms may be less than any assignable solid.

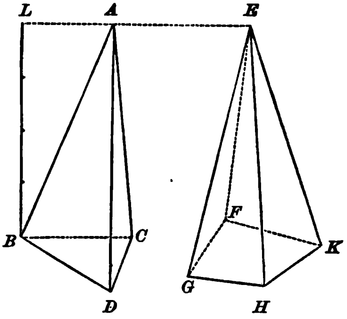
Scholium.—Hence, as we increase the number of sections, the sums of the inscribed and of the circumscribed prisms approach more nearly to each other and to the pyramid. The pyramid is the *limit* to which they approach as the number of sections is indefinitely increased. The difference between them and the pyramid may be made less than any given quantity. As we increase the number of sections to infinity, the difference between the prisms and the pyramid becomes less than any finite quantity, or *zero*.

Proposition 15.

***Theorem.**—Pyramids on equal bases and of the same altitude are equal to each other.*

Let A-BCD and E-$FGHK$ be two pyramids having equal bases, BCD and $FGHK$, and the same altitude, BL. They are equal to each other.

Divide the altitude BL into any number of equal portions, and through the points of division suppose planes to pass parallel to the bases. They will cut equal sections from the pyramids. On these sections inscribe in each of the pyramids a series of prisms. They will be equal to each other (VIII. 9, Cor.), and this will be true whatever the number of divisions in BL. It is therefore true when that number is infinitely increased, and the series of prisms coincide with the pyramids (VIII. 14, Sch.). Hence the pyramids are equal.

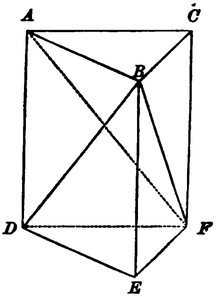


Proposition 16.

***Theorem.**—A triangular prism may be divided into three equal triangular pyramids.*

Let ABC-DEF be a prism; it may be divided into three equal triangular pyramids.

Draw the diagonals *BD, BF, AF*. Through *BD, BF* pass a plane, and also through *BF, AF*. The three pyramids formed will be equal to one another.



The pyramid of which *B* is the vertex and *DEF* the base is equal to the pyramid of which *F* is the vertex and *ABC* is the base, because they are upon equal bases and of the same altitude (VIII. 15). The pyramid of which *F* is the vertex and *ABC* the base may be considered as a pyramid of which *B* is the vertex and *ACF* the base, and is therefore equal to the pyramid of which *B* is the vertex and *ADF* the base (VIII. 15). Hence the three pyramids are equal to one another, and together they make up the prism.

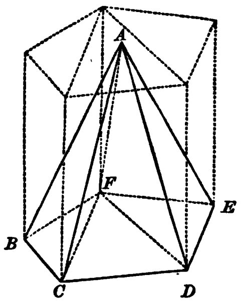
***Corollary.**—The volume of any triangular pyramid is equal to one third the product of the area of the base and the altitude.*

Proposition 17.

***Theorem.**—The volume of a pyramid is equal to one third the product of the area of the base and the altitude.*

Let *A-BCDEF* be a pyramid; its volume is equal to one-third the product of the area of the base, *BCDEF*, and the altitude.

On the base of the pyramid construct a prism of the same altitude as the pyramid, and divide prism and pyramid into triangular prisms and pyramids by planes passing through *CF, DF*, the diagonals of the base.



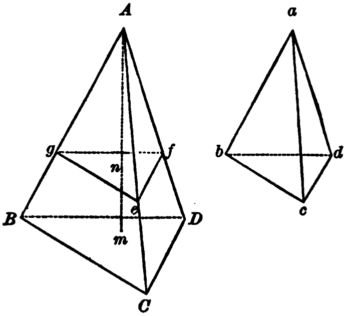
The volume of each of these pyramids is equal (VIII. 16) to one third the volume of the prism on the same base and altitude. Hence the volume of the whole pyramid is one third the volume of the whole prism. But the volume of the prism is equal (VIII. 9) to the product of the area of its base and its altitude. Hence the volume of the pyramid is equal to one third the product of the area of the base and the altitude.

Proposition 18.

Theorem.—*Similar pyramids are to one another as the cubes of their homologous sides.*

Let $A\text{-}BCD$, $a\text{-}bcd$ be two similar pyramids,

then $A\text{-}BCD : a\text{-}bcd :: BC^3 : bc^3$.



Let the pyramid $a\text{-}bcd$ be applied to the pyramid $A\text{-}BCD$, so that a shall be on A, and the line ab on the line AB; b will take some position on AB as g; also because the pyramids are similar, the plane angles at A and a are equal (VIII., Def. 12) to each other, and therefore the planes are equally inclined (VII. 21).

Hence the faces of $a\text{-}bcd$ will coincide with the faces of $A\text{-}BCD$, and, because the angles at the bases are equal, the lines ge, BC, etc. etc. will be parallel, and the plane gef will be parallel to the plane BCD. Let Am, An be the altitudes of the two prisms.

Now (V. 4), and alternately,

$BC : ge :: AB : Ag ::$ (VIII. 12) $Am : An$,

or (IV. 3, Cor.), $\frac{1}{3}Am : \frac{1}{3}An :: BC : ge$;

also (V. 19), $BCD : gef :: BC^2 : ge^2$.

Multiplying together the numerical units of these proportions, term by term,

$$BCD\times\tfrac{1}{3}Am : gef\times\tfrac{1}{3}An :: BC^3 : ge^3.$$

Now, the first terms of this proportion represent (VIII. 17) the volumes of the pyramids. Therefore,

$$A\text{-}BCD : a\text{-}bcd :: BC^3 : bc^3.$$

***Corollary 1.**—Similar pyramids are proportional to the cubes of their altitudes.*

***Corollary 2.**—Similar prisms are proportional to the cubes of their homologous sides.*

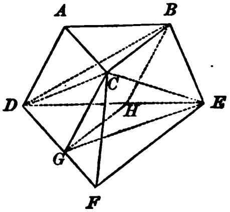
For they would be three times the volume of prisms of the same base and altitude, and therefore in the same proportion to each other.

***Corollary 3.**—Similar polyedrons are proportional to the cubes of their homologous sides or of any homologous lines.*

For they may be divided into the same number of similar triangular pyramids.

Proposition 19.

***Theorem.**—The volume of a frustum of a pyramid is equal to the volume of three pyramids, whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*



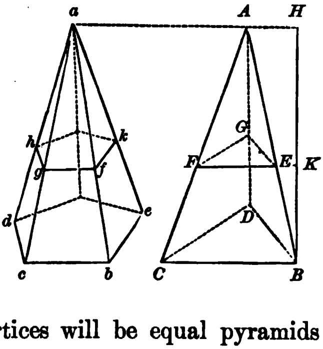
1. Consider the triangular frustum *ABC-DEF*. Divide it into three portions by planes through the diagonals *CD*, *CE* and *CD*, *DB*. One of these will be the pyramid of which *C* is the vertex and *DEF* the base, having for its altitude the altitude of the frustum. Another is the pyramid of which *D* is the vertex and *ABC* the base, also having for its altitude the

altitude of the frustum. The remaining portion is the pyramid having *C* for its vertex, and *DBE* for its base.

From *C* draw *CG* in the plane *AF* parallel to *AD*, meeting *DF* in *G*; from *G* draw *GH* in the plane *DEF*, parallel to *EF* or *CB*; join *BH*, *EG*. Then the triangle *DGH* will be equal to *ABC*, and *DGE* will be a mean proportional between *ABC* and *DEF* (V. 22). Now, the pyramid *C-DBE* is equal to the pyramid *G-DBE* (VIII. 15). But the pyramid *G-DBE* is the same as the pyramid *B-DGE*, which has for its altitude the altitude of the frustum, and for its base a mean proportional between the bases of the frustum.

Hence, the three portions into which the frustum has been divided are equal to three pyramids having a common altitude, which is the altitude of the frustum, and having for bases the two bases of the frustum and a mean proportional between them.

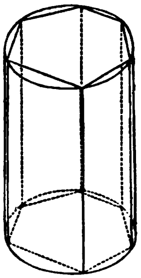
2. The same is true for any frustum of a pyramid.



Let *a-bcde* be a pyramid, and *A-BCD* a triangular one having an equivalent base and the same altitude. They are equal (VIII. 15.) If we cut sections from them at equal distances from their bases, by planes parallel to their bases, the sections will be equal in area; hence, the portions cut off toward the vertices will be equal pyramids (VIII. 15); therefore the remaining frusta are equal. But the volume of the triangular frustum is equal to the volumes of three pyramids having for a common altitude the altitude of the frustum, and for bases the two bases of the frustum and a mean proportional between them. Hence, the same rule is true for the other.

Proposition 20.

Theorem.*—The area of the convex surface of a cylinder is equal to the circumference of the base multiplied by its altitude; and its volume is equal to the area of the base multiplied by its altitude.*



Inscribe in the cylinder a prism whose base is a regular polygon. The area of the lateral surface of this is equal (VIII. 10) to the perimeter of the base multiplied by its altitude, and the volume is equal (VIII. 9) to the area of the base multiplied by the altitude. These being true whatever the number of sides, they are true when that number becomes infinite. In that case the perimeter of the polygon coincides (VI. 10, Cor.) with the circumference of the base, and the prism with the cylinder. Hence the truth of the theorem.

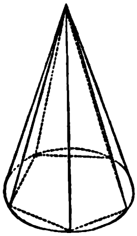
Scholium.—If R = radius of the base of the cylinder; H, its altitude; S, its convex surface; and V, its volume, then the theorem is expressed by the following formulæ:

$$S = 2\pi R \times H;$$
$$V = \pi R^2 \times H.$$

Proposition 21.

Theorem.*—The area of the convex surface of a cone is equal to one half the product of the circumference of the base and the slant height; and its volume is equal to one third the product of the area of the base and the altitude.*

Inscribe in the cone a pyramid whose base is a regular polygon. The area of the lateral surface of this is equal (VIII. 11) to one half the product of the perimeter of the base and slant height, and its volume is equal (VIII. 17) to one third the product of the area of the base and the altitude. These being true whatever the number of the sides, they are true when that number becomes infinite; in which case the perimeter of the polygon coincides (VI. 10, Cor.) with the circumference of the base of the cone, and the prism with the cone. Hence the truth of the theorem.



Scholium.—Let R = radius of base, H = altitude, H' = slant height, S = convex surface, V = volume.

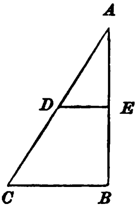
Then, $S = 2\pi R \times \frac{1}{2}H' = 2\pi R \times \frac{1}{2}\sqrt{R^2 + H^2} = \pi R\sqrt{R^2 + H^2}$,

and $V = \pi R^2 \times \frac{1}{3}H.$

Corollary 1.—*The convex surface of a cone is equal to the hypotenuse of the triangle which forms it, multiplied by the circumference of the circle described by its middle point.*

Let AB be the axis, AC the hypotenuse, D its middle point. Then (last Scholium),

$S = \frac{1}{2}AC \times 2\pi CB = AC \times 2\pi(\frac{1}{2}CB) = AC \times 2\pi DE.$

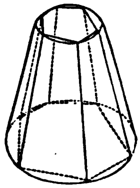


Corollary 2.—*A cone is one third of a cylinder on the same base and of the same altitude.*

Proposition 22.

Theorem.—*The area of the convex surface of the frustum of a cone is equal to the sum of the circumferences of the two bases multiplied by one half the slant height; and its volume is equal to the volume of three cones, whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.*

Inscribe in the frustum of the cone a frustum of a pyramid whose bases are similar regular polygons. The area of the lateral surface of this is equal (VIII. 11, Sch.) to the sum of the perimeters of the two ends multiplied by one half the slant height; and its volume is equal (VIII. 19) to the volume of three pyramids whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them. These being true whatever the number of sides, they are true when that number becomes infinite. In that case the perimeters of the polygons coincide (VI. 10, Cor.) with the circumferences of the bases, the frustum of the pyramid with the frustum of the cone, and the three pyramids into which the frustum is divided with three cones. Hence the truth of the theorem.



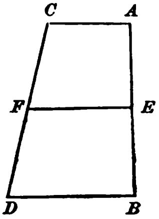
Scholium.—If R, R' be the radii of the bases of the frustum, H its altitude, H' its slant height, S its surface, and V its volume, then

$$S=(2\pi R+2\pi R')\tfrac{1}{2}H'=\pi H'(R+R'),$$

$$V=\tfrac{1}{3}H(\pi R^2+\pi R'^2+\sqrt{\pi R^2\times\pi R'^2}$$

$$=\tfrac{1}{3}\pi H(R^2+R'^2+RR').$$

Corollary.—*The surface of a frustum is equal to the slant height, multiplied by the circumference of a circle described by its middle point.*



Let AB be a portion of the axis, CD the slant height, and F its middle point.

Now (last scholium),

$$S=\tfrac{1}{2}CD\times 2\pi(AC+BD)=CD\times 2\pi\tfrac{1}{2}(AC+BD)$$

$$=CD\times 2\pi FE.$$

Proposition 23.

Theorem.—*The surfaces of similar polyedrons are proportional to the squares of their homologous edges or of any homologous lines.*

Polyedrons are similar when they are composed of the same number of similar faces; hence the whole surfaces are proportional to any two homologous faces; but these faces are proportional to the squares of any two homologous sides; therefore the whole surfaces are proportional to the squares of any two homologous edges, or of any lines proportional to them.

Corollary 1.—*The surfaces of cylinders and cones are proportional to the squares of the radii of their bases or of their altitudes.*

Corollary 2.—*The surfaces of all similar solids are proportional to the squares of their homologous lines, and their volumes to the cubes of their homologous lines.*

The last statement is evident from (VIII. 18, Cor. 3).

Proposition 24.

Theorem.—*Only five regular polyedrons are possible.*

A regular polyedron is one whose faces are equal regular polygons. The angles constituting any solid angle must be less (VII. 20) than four right angles, and there must be at least three plane angles at every solid angle.

The simplest regular polygon is the equilateral triangle. Three angles of an equilateral triangle will form an angle of a tetraedron; four of an octaedron; and five of an icosaedron; six would together be equal to four right angles, and would not form a solid angle.

The regular polygon of four sides is the square. Three angles of a square would form an angle of a cube or hexaedron; four would be equal to four right angles.

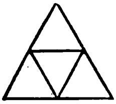
Three angles of a regular pentagon would form an angle of a dodecaedron; four would be greater than four right angles.

Figures of more than five sides cannot form regular polyedrons, for three angles of a hexagon would be equal, and of a heptagon, etc. greater than four right angles.

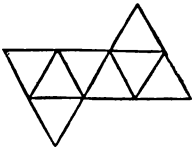
The following gives the above in tabular form:

Name.	No. of faces.	Name of face.
Tetraedron...........................	4	Equilateral Triangle.
Octaedron...........................	8	" "
Icosaedron	20	" "
Hexaedron...........................	6	Square.
Dodecaedron........................	12	Regular Pentagon.

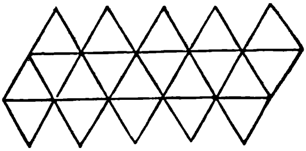
If the following figures be made on pasteboard, the lines cut half through, and the pieces doubled on each other, the regular polyedrons will be formed:



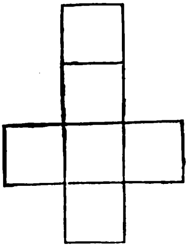
Tetraedron.



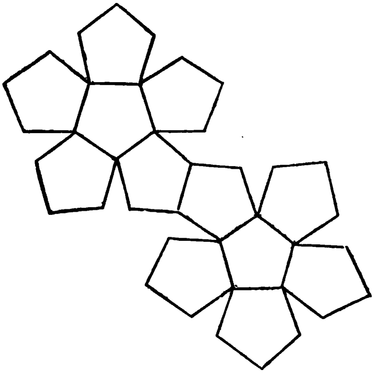
Octaedron.



Icosaedron.



Hexaedron.



Dodecaedron.

EXERCISES.

1. Parallel sections of a prism are equal polygons.

2. The diagonals of a parallelopiped bisect one another.

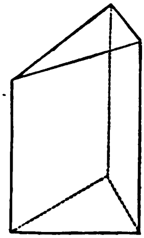
3. Two square prisms are of equal altitude; a side of the base of one is twice a side of the base of the other; are they similar?

4. The cell of a honey-bee is a hexagonal prism; can a number of equal cells exactly fit into each other? What is the lateral surface of a cell, each of whose sides is 1 inch long and $\frac{3}{20}$ of an inch wide? What is the lateral surface of a square cell of the same length and same volume?

5. The sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

6. The volume of a right truncated* triangular prism is equal to the product of its base and one third the sum of its lateral edges.

Divide the solid into three pyramids, as in the case of the frustum of a pyramid.



7. The lateral surface of a regular pyramid is greater than its base.

8. If a cylinder be cut by a plane through an element, the section is a parallelogram.

9. If a cone be cut by a plane through an element, the section is a triangle.

10. If a cone or cylinder be cut by a plane parallel to the base, the section is a circle.

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* A truncated prism is one cut off by a plane oblique to the base.

11. Two similar cylinders are on circles 2 and 3 feet in diameter; what is their relative magnitude? What is the relative magnitude of similar cones on the same bases? If the altitude of one cone be 10 feet, what is the altitude of the other?

12. Suppose that two men be similar solids, one 5, the other 6 feet high; what are the relative amounts of cloth required to cover them? What are their relative weights?

13. Two similar bottles have the diameters of their bases in the proportion of 1 to 2; what is the ratio of their surfaces? of their contents?

14. A cone is cut in two by a plane parallel to the base through the middle point of the altitude; what portion of the volume is cut off toward the vertex?

15. A cylinder is bent around so as to form a circular ring; if we know the inner diameter and thickness of the ring, deduce a rule for finding its volume.

16. To find the ratio of the volumes of two cylinders whose convex areas are equal.

This and the following examples may best be worked out algebraically by using the formulæ for volumes and areas.

17. To find the ratio of the convex areas of two cylinders whose volumes are equal.

18. To find the ratio of the volumes of two cylinders generated by successively revolving a rectangle about its two adjacent sides.

19. What is the convex surface and the volume of a cone, the section of which, through the axis, is an equilateral triangle, whose side is unity? What is the value of this side, for the total surface to be a square metre?

BOOK IX.

SPHERES.

DEFINITIONS.

1. A **sphere** is a solid generated by the revolution of a semicircle about its diameter.

***Corollary.**—The centre of the semicircle is also the centre of the sphere; hence, any point on the surface is equally distant from the centre.*

2. The distance from the centre to the surface is called a **radius**, and a line through the centre terminating both ways in the surface is called a **diameter**.

***Corollary.**—Every diameter being equal to two radii, all diameters of a sphere are equal.*

3. It will be proved that every section of a sphere by a plane is a circle. If the plane pass through the centre of the sphere, the circle is called a **great circle**; if not, a **small circle.**

***Corollary.**—The diameter of a great circle is also the diameter of the sphere; and the intersection of the planes of two great circles is a diameter of the sphere, for both planes pass through the centre.*

4. A line through the centre of a circle of a sphere perpendicular to its plane is called its **axis**: and the intersections of

this line with the spherical surface are the **poles** of the circle.

5. A **spherical triangle** is the space enclosed on the surface of a sphere by the arcs of three great circles.

6. A **spherical polygon** is the space enclosed on the surface of a sphere by arcs of any number of great circles.

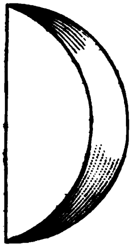
7. A **spherical angle** is the angle between two tangents to the arcs of the great circles which form the angle, at the point where they meet.

8. A **spherical sector** is a solid formed by the revolution of a circular sector about an extreme radius.

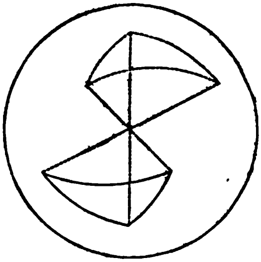
9. A **spherical segment** is the portion of a sphere cut off by a plane.

10. A **spherical zone** is the surface of a sphere between two parallel planes. Its **altitude** is the perpendicular distance between the planes.

11. A **lune** is the portion of the surface of a sphere between the semi-circumferences of two great circles.



12. **Symmetrical spherical triangles** or **polygons** are those formed by the arcs of the same great circles on opposite sides of the sphere.

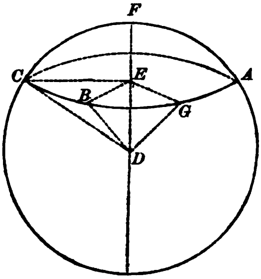


Proposition 1.

Theorem.*—Every section of a sphere made by a plane is a circle.*

Let ABC be any section of a sphere made by a plane; it is a circle.

Take D, the centre of the sphere, and draw (VII. 10) DF perpendicular to the plane of ABC; also join D with any three points of the curve ABC, as G, B, C.



From E, where DF meets the plane, draw EC, EB, EG. The angles DEC, DEB, DEG are right angles; also DB, DC, DG are equal, because they are radii of the sphere, and DE is common to the triangles DEB, DEC, DEG. Therefore (I. 42, Cor. 3) EC, EB, EG are equal; and the same is true of all lines drawn from E to the circumference ABC. Therefore ABC is a circle, and E is its centre.

Corollary 1.*—The pole of a circle is equally distant from all points in it.*

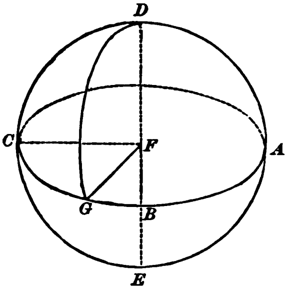
For if CF, GF be joined, we may prove in the triangles CEF, GEF that the chords CF, GF are equal. Hence the arcs CF, GF are equal, and the same may be proved of any arcs from F to ABC.

Corollary 2.*—Any circle made by a plane parallel to ABC would have the same axis and pole.*

Proposition 2.

Theorem.*—A pole of a great circle is at the distance of a quadrant from any point of the circle; and, conversely, if arcs of two great circles between any point on the surface of a sphere and another great circle be quadrants, that point is the pole of the great circle.*

Let *ABC* be a great circle, *DFE* its axis, and *D* and *E* its poles; then *D* and *E* are at the distance of a quadrant from any point of *ABC*.

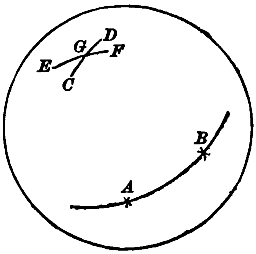


Because *DFE* is the axis of *ABC*, it passes through (IX., Def. 4) its centre, and is at right angles to its plane; and because *ABC* is a great circle, its centre is the centre of the sphere (IX., Def. 3, Cor.); hence *DFC*, *EFC* are right angles, and are measured by the arcs *DC* and *EC*. Therefore *DC* and *EC* are quadrants, and the same may be proved of any other arcs from *D* and *E* to *ABC*.

Conversely, if *DC*, *DG* be quadrants, *D* is the pole of *ABC*. Find *F*, the centre of *ABC*; it will also be the centre of the sphere (IX., Def. 3, Cor.). Join *CF*, *GF*. Because *DC*, *DG* are quadrants, the angles *DFC*, *DFG* are right angles; therefore *DF* is at right angles to the plane *ABC* (VII. 4), at its middle point. It is therefore an axis, and *D* the pole of *ABC* (IX., Def. 4).

Proposition 3.

Problem.*—To pass an arc of a great circle through any two points on the sphere.*



Let *A* and *B* be two points on the surface of the sphere; it is required to pass through them an arc of a great circle.

With the centre *A*, and at a quadrant's distance from it, describe an arc of a great circle, *EF*; with the centre *B*, and at a quadrant's distance from it, describe an arc of a great circle CD; with the centre G, where these arcs cut, and at a quadrant's distance from it, describe the arc of a great circle AB. G will evidently be the pole of AB.

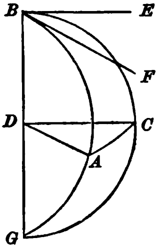
Scholium.—These arcs may be described on a spherical blackboard by a string and chalk, or a wire bent so as to fit the curvature of the sphere.

Proposition 4.

Theorem.—A spherical angle is measured by the arc of the great circle, of which the angular point is the pole, intercepted between the arcs which form the angle.

Let ABC be a spherical angle, and AC an arc of the great circle, of which B is the pole; AC measures the angle B.

The angle ABC is the same as the angle FBE; FB, BE being tangents to AB and BC (IX., Def. 7). Because FB, BE are tangents, they are both at right angles to DB, the common radius; and because B is the pole of the arc AC, BD is also at right angles to AD, DC in the plane ADC (IX., Def. 4); but AD and BF are in the same plane, as also CD and BE; they are therefore parallel, and the angle ADC is equal to the angle FBE (VII. 9). Hence ADC also represents the spherical angle; but ADC is measured by the arc AC (V. 25, Sch.); therefore the spherical angle is measured by the arc AC.



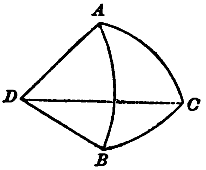
Corollary.—A spherical angle is the inclination of the planes that form the angle.

For ADC represents the angle between the planes ABG, CBG.

Proposition 5.

***Theorem.**—Any two sides of a spherical triangle are together greater than the third.*

Let ABC be a spherical triangle; any two sides are together greater than the third.

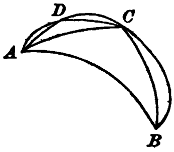


Take D, the centre of the sphere, and join DA, DB, DC. The angle ADB is measured (V. 25, Sch.) by AB, ADC by AC, and CDB by CB. But any two of the three angles which form the solid angle at D are greater (VII. 19) than the third. Hence any two of the arcs of the triangle are greater than the third.

Scholium.—The sides of spherical triangles, being all arcs of circles about the centre of the sphere, are measured in degrees, minutes and seconds.

Proposition 6.

***Theorem.**—The shortest line on the surface of a sphere between two points is the arc of the great circle which joins them.*



Let A and B be two points on the surface of a sphere. The shortest line between them on that surface is the arc AB of a great circle which joins them. For, take any other course, as ACB; let C be any point on ACB, and join (IX. 3) AC, BC by arcs of great circles. Then

$$\text{(IX. 5)}\quad AB < AC + CB.$$

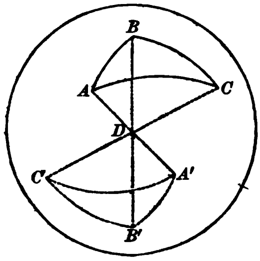
In the same way it may be shown that $AC < AD + DC$, and so on. Hence, AB is less than the sum of AD, DC, etc.; and if these points be taken near enough each other, their sum may be made to differ from ACB by a distance less than any given distance. Therefore AB is less than ACB.

Scholium.—The shortest line between two points on the earth directly east and west from each other is not on the parallel of latitude which joins them. The arc of a great circle deviates to the north of this parallel in the northern, and to the south in the southern hemisphere. The student may test this by stretching a string on a globe.

Proposition 7.

Theorem.—The sides and angles of symmetrical triangles are respectively equal.

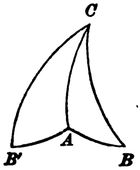
Let ABC be a spherical triangle; produce the radii through its angles beyond the centre to meet the sphere in $A'B'C'$; then the sides and angles of ABC, $A'B'C'$ are respectively equal.



Because the angles ADB, $A'DB'$ are equal (I. 17), the arcs AB, $A'B'$ which subtend them are equal. Similarly, AC is equal to $A'C'$, and BC to $B'C'$. Also, because the three plane angles which make up the two opposite solid angles at D are equal, the planes of one have the same inclination (VII. 21) to each other as the planes of the other; but these inclinations represent (IX. 4, Cor.) the spherical angles of the two triangles; therefore the sides and angles of ABC are respectively equal to the sides and angles of $A'B'C'$.

Corollary.—Isosceles spherical triangles, which have the equal sides and included angle of each equal, are equal in all their parts.

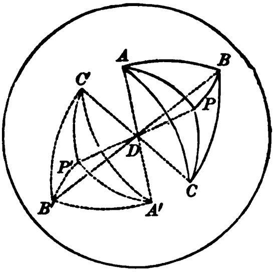
If we place the two triangles in last figure so as to have one side, AC, common, they are still called symmetrical triangles; but they cannot be applied to each other unless $CB' = CA = CB$; that is, unless they are isosceles. The student may see this by constructing symmetrical triangles on a spherical blackboard. Though equal, they cannot be placed on each other so as to fit, the order of arrangements of the parts being the reverse.



Proposition 8.

Theorem.—Symmetrical triangles are equal in area.

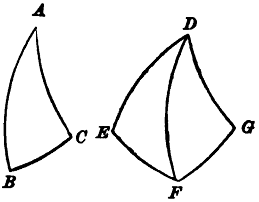
Let ABC, $A'B'C'$ be two symmetrical triangles; they are equal in area.



Suppose a small circle of the sphere to be passed through the points A, B and C, and let P be its pole. Join P with D, the centre of the sphere, and produce it to meet the surface of the sphere in P'; join (IX. 3) P with A, B, C, and P' with A', B', C', by arcs of great circles. Now, because PA, PB, PC are equal (IX. 1, Cor. 2), and equal to $P'A'$, $P'B'$, $P'C'$, these latter are also equal to each other. Hence, the triangles PAB, $P'A'B'$ are isosceles, and they have the two sides and included angle of each equal; hence they are equal (IX. 7, Cor.) in area. In the same way, PAC, $P'A'C'$, and also PBC, $P'B'C'$, are equal to each other; therefore the whole triangle ABC is equal to the whole triangle $A'B'C'$. In case P and P' fell without the triangles, we would add two triangles and subtract the third; the remainder would be the triangles ABC, $A'B'C'$.

Proposition 9.

Theorem—*If two triangles, on the same or equal spheres, have three sides of one respectively equal to three sides of another, the triangles will be equal in all their parts.*



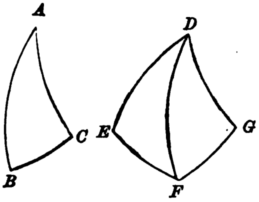
Let *ABC*, *DEF* be two triangles which have the sides of one respectively equal to the sides of the other; they are equal in all their parts.

Let *ABC* be applied to *DEF*, or, if the order of arrangement of the parts be the reverse, to the symmetrical triangle *DFG*; then, as in (I. 4), it may be proved equal to *DEF* or *DFG*. But *DEF* is equal to *DFG* (IX. 8). Hence, *ABC* is equal in either case in all its parts to *DEF*.

Proposition 10.

Theorem.—*If two triangles, on the same or equal spheres, have two sides and the included angle of one respectively equal to the two sides and the included of another, they are equal in all their parts.*

Let *ABC*, *DEF* be two triangles, having *BA*, *AC* and the angle *BAC* of one respectively equal to *ED*, *DF* and the angle *EDF* of the other; they will be equal in all their parts.



Let *ABC* be applied to the triangle *DEF* (or to the equal symmetrical triangle *DFG*), as in (I. 7). The proof of the theorem will be precisely as in the similar case of plane triangles.

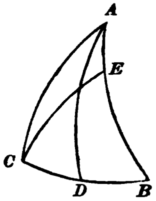
Scholium.—The various *Problems* of Book I., from the

1st to the 12th inclusive, may be applied to spherical triangles, etc., as well as to plane, and proved in a similar manner; all lines drawn must, however, be arcs of great circles.

Proposition 11.

Theorem.*—In an isosceles spherical triangle the angles opposite to the equal sides are equal; and, conversely, if a triangle have two angles equal to each other the opposite sides are also equal.*

Let ABC be an isosceles spherical triangle having AB equal to AC; the angle ABC is also equal to the angle ACB.



Bisect the arc BC in D, and pass (IX. 3) from A to the middle of the base D an arc of a great circle. Then the two triangles ADB, ADC, having the three sides of one equal to the three sides of the other, are equal (IX. 9) in all their parts; hence the angle ABD is equal to the angle ACD.

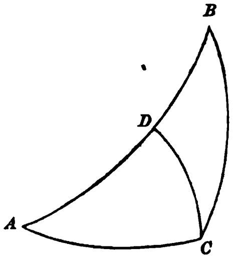
Conversely, if the angle ABC be equal to the angle ACB, the side AB is equal to the side AC. For if AC be not equal to AB, suppose that EC is equal to AB. Pass through B and E an arc of a great circle; then the triangles ABC, ECB have AB, BC and the included angle ABC, respectively equal to EC, CB and the included angle ECB. The triangles are therefore equal (IX. 10), which is impossible. Hence AC is not unequal to AB; that is, it is equal to it.

Proposition 12.

Theorem.*—The greater angle of every spherical triangle has the greater side opposite to it; and, conversely, the greater side has the greater angle opposite to it.*

Let ABC be a spherical triangle having the angle ACB greater than the angle ABC; AB is greater than AC.

Make $BCD = DBC$, then (IX. 11) $DB = DC$; hence $AB = AD + DC$. But (IX. 5) $AD + DC > AC$; therefore $AB > AC$.

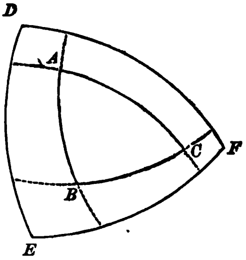


Conversely, if $AB > AC$, then $ACB > ABC$. It cannot be equal, because AB would be equal (IX. 11) to AC; nor less, for then AB would be less than AC, by this theorem; but these are both contrary to the hypothesis, hence $ACB > ABC$.

Proposition 13.

Theorem.—If the angular points of a spherical triangle be made the poles of the sides of another triangle, the angular points of the second triangle will be the poles of the sides of the first.

Let ABC be a spherical triangle, and let A be the pole of EF, B the pole of DF, and C the pole of ED; then will D, E and F be the poles of BC, AC and AB, respectively.



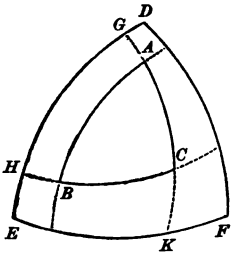
Because A is the pole of EF, the great circle between A and E is a quadrant (IX. 2), and because C is the pole of ED, the great circle between C and E is a quadrant; and the arcs of two great circles, drawn from E to two points of AC, are quadrants; hence, E is the pole of AC (IX. 2). In the same manner it may be proved that D is the pole of BC, and F of AB.

Definition.—Triangles such as the above are called *polar* triangles; also *supplemental* triangles, from a property proved in the next proposition.

Proposition 14.

***Theorem.**—Any angle of a spherical triangle is the supplement* of the side opposite in the polar triangle.*

Let ABC be a spherical triangle, and DEF its polar triangle. The angles of ABC are supplements of the opposite sides of DEF, and the angles of DEF are supplements of the opposite sides of ABC.



Produce, if necessary, the sides of ABC to meet the sides of DEF. Now, because E is the pole of the arc AC, EG is a quadrant; and because D is the pole of BC, DH is a quadrant;

therefore $EG+DH=$ a semicircle,

or $ED+GH=$ a semicircle;

hence, GH is the supplement of ED; but GH measures the angle C (IX. 4); therefore DE is the supplement of C; and in the same way it may be proved that EF is the supplement of A, and DF of B.

Also, because CG and AK are quadrants,

$$AC+GK=\text{a semicircle};$$

hence AC is the supplement of GK; but GK measures E; therefore AC is the supplement of E; similarly, BC is the supplement of D, and AB of F.

***Corollary.**—Any angle of a spherical triangle, together with the opposite side of its polar triangle, is equal to* 180°.

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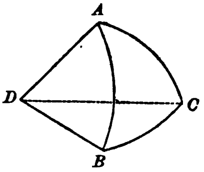
* The supplement of an arc or angle is what it differs from a semicircle, or 180°.

Proposition 15.

Theorem.*—The three sides of a spherical triangle are together less than the circumference of a great circle.*



Let ABC be a spherical triangle; its three sides together are less than the circumference of a circle.

Take D, the centre of the sphere, and join DA, DB, DC. Then the angles which form the solid angle D are measured (V. 25, Sch.) by the arcs which form the three sides of the triangle; but these angles are together less (VII. 20) than four right angles; therefore the arcs are together less than the circumference of a great circle.

Scholium.—For the same reasons the sum of the sides of a spherical polygon of any number of sides, is less than the circumference of a great circle.

Proposition 16.

Theorem.*—The sum of the angles of a spherical triangle are together greater than two, and less than six right angles.*

For an angle of a triangle, together with the opposite side of the polar triangle, is equal to 180° (IX. 14, Cor.); hence, the three angles of any triangle and the three sides of the polar triangle are equal to 540°, or six right angles. But the sides of the polar triangle are together greater than 0°, and less than 360° (IX. 15). Hence, the angles of the triangle are less than 540°, or six right angles, and greater than 180°, or two right angles.

Corollary.*—A spherical triangle may have two or three right angles.*

Proposition 17.

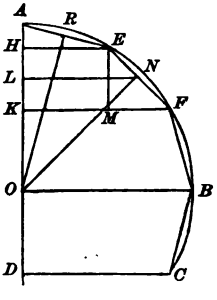
Theorem.—*If two spherical triangles, on the same or equal spheres, have three angles of one respectively equal to three angles of the other, the triangles will be equal in all their parts.*

For, if the angles of one be equal to the angles of the other, the sides of their respective polar triangles shall be equal, each to each. Hence, the angles of the polar triangles shall be equal, each to each (IX. 9). Therefore the sides of the original triangles shall be equal, each to each. Hence, the triangles are every way equal (IX. 9).

Scholium.—This is different from the case of plane triangles, which are not mutually equilateral if mutually equiangular. But it is only true for triangles on the same or equal spheres. *Any* two spherical triangles are not mutually equilateral if mutually equiangular.

Proposition 18.

Theorem.—*The area of a spherical zone of one base is equal to the circumference of a great circle of the sphere multiplied by the altitude of the zone.*



Let *ABC* be an arc of a circle which, by its revolution about *AD*, describes a spherical zone; the area of the zone is equal to a circumference of a great circle of the sphere, multiplied by *AD*.

Let *AE*, *EF*, etc., be equal lines inscribed in the arc, of such a length that the last one, *BC*, will terminate at the extremity of the arc; and suppose them to revolve about *AD* with the arc; they will generate cones or frusta of cones. Draw *EH*, *FK*, *BO*, *CD* perpendicular to *AD*. From *N*, the middle point of *EF*, draw NL parallel to EH, and draw NO to the centre, and from E, EM parallel to AD.

Now, because EMF, NLO are similar (V. 7) triangles,

(V. 4), $EF : EM :: ON : NL :: 2\pi ON : 2\pi NL.$

Hence (V. 14), $EF \,.\, 2\pi NL = EM(\text{or } HK)\, 2\pi ON.$

But (VIII. 22, Cor.), $EF \,.\, 2\pi NL = \text{surf. } EF;$

$\therefore$ Surf. $EF = 2\pi ON \times HK.$

In a similar manner it can be proved from (VIII. 21, Cor. 1),

$$\text{Surf. } AE = 2\pi OR(\text{or } ON) \times AH;$$

also, surf. $BF = 2\pi ON \times OK,$

and also of the others. Adding these,

Surf. $(AE + EF +, \text{etc.}) = 2\pi ON(AH + HK +, \text{etc.})2\pi = ON.AD.$

Now, suppose the number of inscribed sides to be infinitely increased; they will coincide with the arc of the circle; ON will become the radius of the circle, and $2\pi ON$ the circumference of a great circle. Hence,

$$\text{Surf. } ABC = \text{circ. great circle} \times AD.$$

Corollary 1.*—The area of a spherical zone of two bases may be found by subtracting from each other, two zones of one base each.*

That is, surf. $FC = \text{surf. } ABC - \text{surf. } AF.$

Corollary 2.*—The areas of zones on the same or equal spheres are proportional to their altitudes.*

Proposition 19.

Theorem.*—The surface of a sphere is equal to a circumference of a great circle multiplied by its diameter.*

For a sphere may be considered as a zone generated by a semicircle. Hence, its altitude would be the diameter of the semicircle or of the sphere. But the surface of a zone is equal (IX. 18) to the circumference of a great circle multiplied by its altitude. Hence the theorem.

Corollary 1.—The surface of a sphere is equal to the areas of four great circles.

Let R = radius of sphere, and S = surface;

then (this Prop.), $S = 2\pi R \times 2R = 4\pi R^2$ (VI. 12, Cor. 1).

Corollary 2.—The surfaces of two unequal spheres are proportional to the squares of their radii.

For $S : S' :: 4\pi R^2 : 4\pi R'^2 :: R^2 : R'^2$.

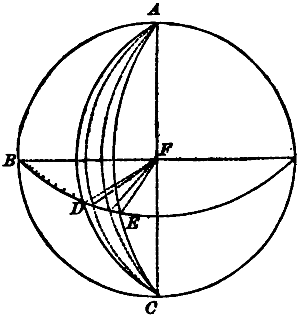
Proposition 20.

Theorem.—On the same or equal spheres the surfaces of lunes are proportional to their angles.

Let *ABCD*, *ADCE* be two lunes (IX., Def. 11); their surfaces are proportional to their angles *DAB*, *EAD*.

1. Suppose the arcs *BD*, *DE*, which measure the angles of the lunes, to be commensurable.

Through the points of division of *BD*, *DE* and through *A* and *F* suppose a number of planes to be passed. These planes will divide the surface of the lunes into equal triangles (IX. 9), and the angles of the lunes at *F* into the same number of equal angles (III. 19).



Hence, Surf. *ABCD* : surf. *ADCE* :: ang. *DFB* : ang. *EFD*.
:: ang. *DAB* : ang. *EAD*.

2. If the arcs are incommensurable, the proof is the same as given in (V. 1).

Corollary 1.—Since a sphere may be considered as a lune of 360°, *we have the proportion*

Surf. lune : surf. sphere :: ang. of lune : four right angles.

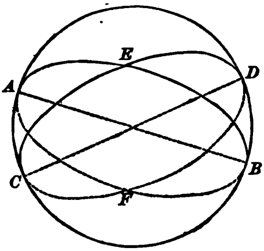
Corollary 2.—If L be the angle of the lune,

then Surf. of lune $= \frac{L}{360} 4\pi R^2 = \frac{L}{90} \pi R^2$.

Proposition 21.

Theorem.—If two great circles be cut by a third, the two triangles formed are together equal to a lune contained by the first two circles.

Let the two great circles $AEBF$, $CEDF$ be cut by the circle $CADB$; the two triangles AEC, DEB are together equal to the lune $EDFD$ (which we will call the lune E).



The two triangles EAC, FDB have the angles E and F equal, also $FD = EC$, because they are both supplements of ED, and $FB = AE$, because they are both supplements of EB. Hence (IX. 9) the triangles are equal. Add to each the triangle EDB, and $AEC + EDB =$ lune E.

Proposition 22.

Theorem.—The area of a spherical triangle is equal to one half the lune whose angle is the excess of the three angles of the triangle over two right angles.

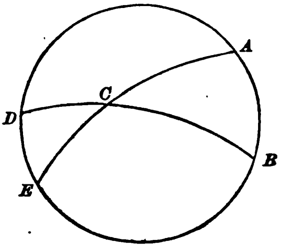
Let ABC be a triangle; its area is equal to

$$\tfrac{1}{2} \text{ lune } (A + B + C - 180°).$$

Complete the circle *ABED*, and produce *AC*, *BC* to meet it in *E* and *D*. Then *BD*, *AE* are semicircles; and the two triangles *ACB*, *DCE* are equal (IX. 21) to the lune *C*; therefore

$$\text{Lune } A + \text{lune } B + \text{lune } C = \text{hemisphere} + 2ABC;$$

the triangle *ABC* being included three times in the first member of this equation. But a hemisphere may be considered as a lune whose angle is 180°.



Hence $2ABC = \text{lune } A + \text{lune } B + \text{lune } C - \text{lune } 180°$,

or $ABC = \frac{1}{2} \text{ lune } (A + B + C - 180°)$.

Corollary.—Since the area of the lune $(A + B + C - 180)$ $= \text{(IX. 20, Cor. 2)} \dfrac{A + B + C - 180°}{360} 4\pi R^2$, if we now make the right angle the unit of angles,

$$ABC = \tfrac{1}{2} \frac{A + B + C - 2}{4} 4\pi R^2$$

$$= (A + B + C - 2) \frac{4\pi R^2}{8}.$$

The excess of the three angles of a triangle over two right angles is called the *spherical excess*, and the one eighth part of the surface of a sphere is called the *tri-rectangular triangle*, because it contains three right angles. Hence the *area of a spherical triangle is equal to the spherical excess multiplied by the tri-rectangular triangle.*

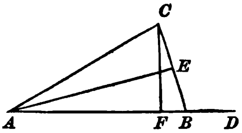
Scholium.—To illustrate, suppose $A = 200°$, $B = 80°$, and $C = 170°$, and $R = 20$,

then $ABC = \left(\dfrac{200 + 80 + 170}{90} - 2\right) 200\pi = 600\pi.$

Proposition 23.

***Theorem.**—The solid generated by the revolution of an acute-angled triangle about a line through its vertex, not cutting the triangle, is equal to the surface described by its base multiplied by one half its altitude.*

Let ABC be an acute-angled triangle; the solid generated by the revolution of ABC about AD is equal to the surface described by BC, multiplied by one half the altitude AE.

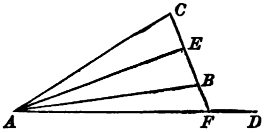


1. Let one side, as AB, lie in the axis AD. Draw CF perpendicular to AB. Then ACB by its revolution will generate two cones having a common base of which CF is the radius, and altitudes AF, FB. Their volumes are

(VIII. 21), $\frac{1}{3}AF\times\pi CF^2$
and $\frac{1}{3}FB\times\pi CF^2$, and their sum is
$\frac{1}{3}AB\times\pi CF^2=\frac{1}{3}\pi CF\times AB\,.\,CF.$

But $AB\,.\,CF$ represents (V. 16, Cor. 2) the double area of the triangle ABC, which is also represented by $CB\,.\,AE$; hence Vol. $ABC=\frac{1}{3}\pi CF.CB\,.AE=\frac{1}{3}AE\,.\pi CF\,.\,CB.$ But $\pi CF.CB$ is (VIII. 21, Cor. 1) the surface generated by CB, and $\frac{1}{3}AE$ is one third the altitude. Hence the theorem.

2. Let the line AB not be in the axis AD. Produce CB to F. Then from last case

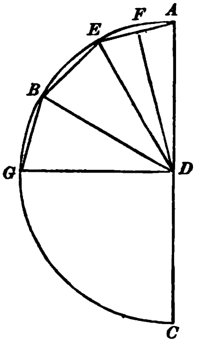


Vol. $ACF=$ surf. $CF\times\frac{1}{3}AE$;
also Vol. $ABF=$ surf. $BF\times\frac{1}{3}AE.$
Hence Vol. $ABC=$ surf. $CB\times\frac{1}{3}AE.$

Proposition 24.

***Theorem.**—The volume of a sphere is equal to its surface, multiplied by one third its radius.*

Let ABC be the semicircle which, by its revolution about AC, describes a sphere; then Vol. $ABC=$ surf. $ABC\times\frac{1}{3}AD.$



In the circle ABC inscribe a regular polygon of such a number of sides that the quadrant AG will be evenly divided by the angles E, B, etc., and suppose this to revolve with the semicircle AC. Let AE, EB be sides of this polygon, and join ED, BD. The sum of the solids formed by the revolution of the various acute-angled triangles ADE, EDB, etc. about AC will be

(IX. 23), Vol. $AED =$ surf. $AE \times \frac{1}{3}DF$,

Vol. $EDB =$ surf. $EB \times \frac{1}{3}DF$, etc. etc.

Adding all these, the sum of the volumes generated by all the triangles will be equal to the sum of the surfaces generated by the sides of the polygon, multiplied by one third the apothem. This being true whatever the number of sides, it is true when the polygon becomes a circle, when the volume will be the volume of the sphere, the surface, the surface of the sphere, and the apothem its radius. Hence the volume of a sphere is equal to its surface, multiplied by one third its radius.

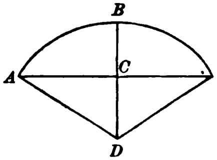
Corollary 1.—If $V =$ volume, $R =$ radius, and $D =$ the diameter,

then (IX. 19, Cor. 1), $V = 4\pi R^2 \times \frac{1}{3}R = \frac{4}{3}\pi R^3$;

and since $R = \frac{1}{2}D$, $R^3 = \frac{1}{8}D^3$;

hence $V = \frac{1}{6}\pi D^3$.

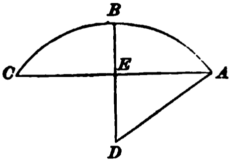
Corollary 2.—*Spheres are to one another as the cubes of their radii, or as the cubes of their diameters.*



Scholium.—A similar demonstration will prove that the volume of a spherical sector is equal to the surface described by the arc multiplied by one third its radius. Its arc, AB, describes a zone which is equal (IX. 18) to $2\pi R \times BC$. Hence the volume of the sector is equal to $\frac{2}{3}\pi R^2 \times BC$.

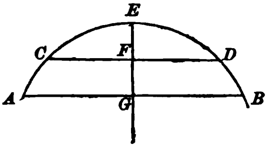
Proposition 25.

***Theorem.**—The volume of a spherical segment of one base is represented by $\pi A^2(R - \frac{1}{3}A)$, R being the radius of the sphere and A the altitude of the segment.*



Let AEB be revolved about BE; it will describe a spherical segment. Let $AD = R$, $EB = A$; then the volume of the segment is $\pi A^2(R - \frac{1}{3}A)$.

For vol. $AEB =$ vol. $ADB -$ vol. AED. Vol. ADB being a spherical sector is equal (IX. 24, Sch.) to $\frac{2}{3}\pi R^2 . A$; and vol. AED being a cone is equal (VIII. 21) to $\pi EA^2 \times \frac{1}{3}ED$; now, $ED = R - A$ and $AE^2 = AD^2 - ED^2 = R^2 - (R - A)^2 = 2AR - A^2$; hence vol. $AED = \pi(2AR - A^2)\frac{1}{3}(R - A) = \frac{1}{3}\pi(2AR^2 - 3A^2R + A^3)$. Subtracting this from the vol. $ADB = \frac{2}{3}\pi R^2 A$, and we have vol. $AEB = \frac{2}{3}\pi R^2 A - \frac{1}{3}\pi(2AR^2 - 3A^2R + A^3) = \pi(A^2R - \frac{1}{3}A^3)$ $= \pi A^2(R - \frac{1}{3}A)$.



Scholium.—The volume of a spherical segment with two bases may be found by subtracting from each other, two of one base each.

Thus, Vol. $ACFG =$ vol. $AEG -$ vol. CEF.

Proposition 26.

***Theorem.**—The surface of a sphere is to the surface of a circumscribing cylinder as 2 is to 3; and their volumes bear the same ratio to each other.*[*]

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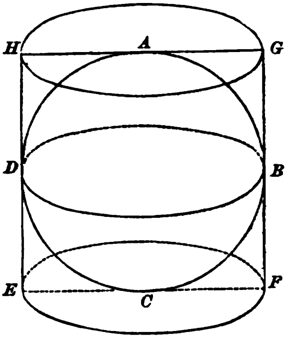
[*] This property was discovered by Archimedes, and the figure was engraved on his tomb, where it was discovered by Cicero when quaestor in Sicily.

Let *ABCD* be a sphere, and *EFGH* a circumscribing cylinder, then



Surf. *ABCD* : surf. *EFGH* : : 2 : 3,
Vol. *ABCD* : vol. *EFGH* : : 2 : 3.

The surface of the sphere is equal (IX. 19, Cor. 1) to $4\pi R^2$, R being the radius of the sphere. The convex surface of the cylinder is (VIII. 20) $2\pi R \times 2R = 4\pi R^2$, and the area of the two ends (VI. 12, Cor. 1) is $2\pi R^2$; hence the whole surface of the cylinder is $6\pi R^2$.

Hence, Surf. *ABCD* : surf. *EFGH* : : $4\pi R^2 : 6\pi R^2$: : 2 : 3.

Also the volume *ABCD* is (IX. 24, Cor. 1) $\frac{4}{3}\pi R^3$, and the volume *EFGH* is (VIII. 20) $\pi R^2 \times 2R = 2\pi R^3$.

Hence, Vol. *ABCD* : vol. *EFGH* : : $\frac{4}{3}\pi R^3 : 2\pi R^3$: : 2 : 3.

EXERCISES.

1. Circles made by planes equidistant from the centre of the sphere are equal.

2. The intersection of two spheres is a circle, whose centre is in a line joining the centres of the spheres.

3. If from the ends of one side of a spherical triangle arcs of great circles be drawn to a point within the triangle, these two arcs shall be less than the other sides of the triangle.

4. In a tri-rectangular triangle each angle is the pole of the opposite side.

5. To draw a tangent plane to a sphere at a given point on the surface.

6. If a cone and a sphere be inscribed in a cylinder, their volumes are as 1, 2, and 3.

7. The volume of a sphere is to the volume of the inscribed cube as is π to $\frac{2}{\sqrt{3}}$.

8. The surface of an inscribed cube is equal to eight times the square of the radius of the sphere.

9. Draw on a spherical blackboard the polar triangle of a triangle with a right angle; with two right angles; with three right angles; with two obtuse angles.

10. A sphere may be inscribed in a regular polyedron.

11. The volume of a regular polyedron is equal to its surface multiplied by one third the radius of the inscribed sphere.

12. Compare the convex surfaces of a sphere and the circumscribed cylinder and cone, the cone having an angle of 60° at the vertex.

13. Compare the volumes of a sphere and its circumscribed cube, cylinder and cone, the cone having an angle of 60° at the vertex.

14. A square prism is inscribed in the cylinder of (IX. 26). Compare the volumes of sphere and prism.

15. Knowing the radii of two parallel sections of a sphere, and their distance from each other, to find the radius of the sphere.

16. If from a point in space we draw secants to a given sphere, the product of the distances from this point to the two points of intersection of each secant, with the sphere, is constant.

MENSURATION OF SOLIDS.

SECTION I.

1. To find the volume of a prism.

Multiply the area of the base by the altitude. (VIII. 9.)

2. To find the surface of a prism.

Multiply the perimeter of the base by the altitude, and add the areas of the two ends. (VIII. 10.)

3. To find the volume of a pyramid.

Multiply the area of the base by one third the altitude. (VIII. 17.)

4. To find the surface of a pyramid.

Multiply the perimeter of the base by one half the slant height, and add the area of the base. (VIII. 11.)

5. To find the volume of a frustum of a pyramid.

Add together the areas of the two ends and a mean proportional between them; multiply the sum by one third the altitude. (VIII. 19.)

6. To find the surface of a frustum of a pyramid.

Add together the perimeters of the two ends; multiply the sum by one half the slant height; to this add the areas of the ends. (VIII. 11, Sch.)

7. To find the volume of a cylinder.

Multiply the area of the base by the altitude. (VIII. 20.)

8. To find the surface of a cylinder.

Multiply the perimeter of the base by the altitude, and add the areas of the two ends. (VIII. 20.)

9. To find the volume of a cone.

Multiply the area of the base by one third the altitude. (VIII. 21.)

10. To find the surface of a cone.

Multiply the perimeter of the base by one half the slant height, and add the area of the base. (VIII. 21.)

11. To find the volume of a frustum of a cone.

Add together the squares of the radii of the ends and their product, and multiply the sum by π, and by one third the altitude of the frustum. (VIII. 22, Sch.)

12. To find the surface of a frustum of a cone.

Add together the radii of the two ends, and multiply the sum by π and by the slant height; to this add the areas of the ends. (VIII. 22, Sch.)

13. To find the surface of a regular polyedron.

I. *Multiply the area of one of the faces by the number of faces.*

II. *Multiply the square of one of the edges by the surface of a similar polyedron whose edge is unity.* (VIII. 23.)

14. To find the volume of a regular polyedron.

I. *Multiply the surface by one sixth the perpendicular distance between the opposite faces, if the faces be opposite; if not, by one third the radius of the inscribed sphere.**

II. *Multiply the cube of one of the edges by the volume of a similar polyedron whose edge is unity.* (VIII. 18, Cor. 3.)

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* This rule is proved by supposing the polyedron to be made up of pyramids whose common vertex is at the centre, and whose bases are the faces of the polyedron.

TABLE OF REGULAR POLYEDRONS WHOSE EDGES ARE UNITY.

NAME.	No. of faces.	Shape of face.	Surface.	Volume.
Tetraedron..........	4	Equilateral triangle.	1.732051	0.117851
Hexaedron (cube).	6	Square.................	6.000000	1.000000
Octaedron...........	8	Equilateral triangle.	3.464102	0.471404
Dodecaedron.......	12	Regular pentagon....	20.645729	7.663119
Icosaedron..........	20	Equilateral triangle.	8.660254	2.181695

15. To find the surface of a sphere.

I. *Multiply the circumference of a great circle by its diameter.* (IX. 19.)

II. *Multiply the square of the radius by* 4π. (IX. 19, Cor. 1.)

16. To find the volume of a sphere.

I. *Multiply its surface by one third the radius.* (IX. 24.)

II. *Multiply the cube of the radius by* $\frac{4}{3}\pi$. (IX. 24, Cor. 1.)

III. *Multiply the cube of the diameter by* $\frac{1}{6}\pi$. (IX. 24, Cor. 1.)

17. To find the surface of a spherical zone.

Multiply the altitude of the zone by the circumference of a great circle of the sphere. (IX. 18.)

18. To find the volume of a spherical segment of one base.

Apply the formula $V = \pi A^2(R - \frac{1}{3}A)$;

R being the radius of the sphere, and A the altitude of the segment. (IX. 25.)

19. To find the volume of a spherical segment of two bases.

Subtract the volumes of two spherical segments of one base. (IX. 25, Sch.)

20. To find the volume of a spherical sector.

Multiply the surface of the zone which forms its base, by one third the radius of the sphere. (IX. 24, Sch.)

21. To find the surface of a cylindrical ring.*

Add the thickness of the ring to the inner diameter; multiply this by the thickness of the ring and by π^2.

22. To find the volume of a cylindrical ring.

Add the thickness of the ring to the inner diameter; multiply the sum by the square of one half the thickness and by π^2.†

23. To find the volume of an irregular solid.

Immerse the solid in water, and measure the rise of the water. If now we know the size of the vessel, we may calculate the volume of displaced water, and consequently the volume of the solid.‡

EXAMPLES.

1. What is the lateral surface of a hexagonal prism, each of the sides of its base being 2 feet and its altitude 8 feet? *Ans.* 96 sq. ft.

2. What are the volume and lateral surface of a cylinder circumscribing the prism of last example? *Ans.* 32π; 32π.

3. The pyramid of Cheops is 764 feet square at the base and 480 feet in perpendicular height. What is its volume? *Ans.* 93391360 cu. ft.

4. If the pyramid of last example were cut off by a plane parallel to its base at a distance of 320 feet from the ground, what would be the volume of the upper portion? *Ans.* $\frac{1}{27}$ of the whole.

---

* A cylindrical ring is formed by bending a cylinder so that its axis is a circle. If R be the inner radius of the ring, and r the radius of the cylinder, the length of the axis is $2\pi(R+r)$, and this, multiplied by the perimeter of the cylinder, $2\pi r$, will be the surface, or,

$S=2\pi(R+r)\times 2\pi r=\pi^2(2R+2r)2r$, which agrees with the rule.

† $V=2\pi(R+r)\times \pi r^2=\pi^2(2R+2r)r^2$.

‡ This is only an approximate value, on account of the porosity of most solids.

5. A conical tent is 10 feet in diameter at the base and 10 feet high. How many square feet of canvas are required to cover it? *Ans.* 175.614 +.

6. A square prism is inscribed in a cylinder whose radius is 6 feet and length 10 feet; what is its volume? *Ans.* 720.

The diameter of the cylinder multiplied by the radius will be the area of the base of the prism.

7. A tin vessel containing a liquid quart is cylindrical, 4 inches in diameter; what is its height? How much tin is required to make it? *Ans.* 4½ nearly; 69.12.

8. A square pyramid 2 feet in altitude has its base inscribed in a circle 1 foot in radius. What would be the volume of a similar pyramid whose base was inscribed in a circle of 9 times the area? *Ans.* 108 cu. ft.

9. A cylindrical log is 20 feet long and 2 feet in radius; how many square feet of inch boards can be cut from it, allowing ¼ for slabs and waste in sawing? *Ans.* 2261.952.

10. Bunker Hill Monument is 220 feet high, 30 feet square at the base and 15 feet at the top. Through it runs a circular opening 15 feet in diameter at the base and 11 feet at the top. How many cubic feet of masonry does it contain? *Ans.* 86068 +.

11. A cubic foot of copper is drawn out into a wire $\frac{1}{16}$ of an inch in diameter. What is the length of the wire? *Ans.* 2¼ miles, nearly.

12. An iron pipe is 3 inches in bore, ½ inch thick, and 20 feet long. Find its weight, supposing that a cubic inch of iron weighs 4.526 oz. *Ans.* 373 lbs.

13. A square iron rod an inch thick weighs 10½ lbs. Find the weight of a round iron rod of the same length and thickness. *Ans.* 8.2467 lbs.

14. The height of a cylinder is to be 10 times the radius of the base, and the volume is to be 25 cubic feet. Find the radius. *Ans.* .927 feet.

15. The volume of a cone is 22½ cubic feet, and the circumference of the base is 9 feet; find the height. *Ans.* 10.47 feet.

16. The shaft of Pompey's Pillar is a single block of granite 90 feet high; the diameter of one end is 9 feet, and of the other end 7½ feet. Find the volume. *Ans.* 4824.24 cu. ft.

17. Every edge of a pyramid on a square base is one foot. What is its volume? *Ans.* $\frac{\sqrt{2}}{6}$ cu. ft.

18. A vessel with a lid in the form of a right cylinder holds 3141.6 cubic inches. Find the area of the internal surface if the height be equal to the radius of the base. *Ans.* 1256.64 sq. in.

19. What is the largest cube that can be cut from a sphere 10 feet in radius? *Ans.* $\frac{8000}{3\sqrt{3}}$.

20. A ball weighing 20 lbs. is 3 inches in diameter. What is the weight of a ball 5 inches in diameter? *Ans.* 92.59 lbs.

21. A sphere has a radius of 6 feet. What are its volume and surface? *Ans.* 288π; 144π.

22. Find how many times the volume of the earth exceeds the volume of the moon, the former being supposed 7900 miles in diameter, and the latter 2160. *Ans.* Nearly 49.

23. A tent is made in the form of a frustum of a cone surmounted by a cone. The diameters of the ends of the frustum are 28 and 16 feet, the height of the frustum 8 feet, and the whole height of the tent 14 feet. How many square yards of canvas are required to cover it?

Ans. 105, nearly.

24. How far from the centre of a sphere 6 feet in diameter must two parallel planes be passed to divide it into 3 zones of equal convex surface? *Ans.* 1 foot.

25. Find the weight of a shell 3½ inches thick, the external diameter of which is 1 foot 5½ inches, composed of metal a cubic foot of which weighs 480 lbs. *Ans.* 611.13 lbs.

26. A sphere is inscribed in a cylinder touching its circumference and ends. A cone is constructed on the same base and of the same altitude as the cylinder. Compare the volumes of the three solids.

Ans. Vol. cone : sphere : cylinder : : 1 : 2 : 3.

27. The altitude of either Frigid Zone is 327 miles, Temperate 2054 miles, Torrid 3150 miles. What is the surface of each? Diameter of the earth = 7912.

Ans.
Frigid, 8,128,252 sq. miles.
Temperate, 51,056,587 " "
Torrid, 78,293,218 " "

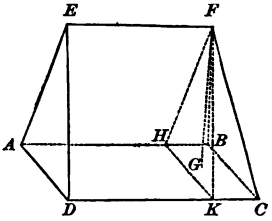
28. How far above the earth must a person ascend to see one third its surface? *Ans.* Diameter of the earth.

SECTION II.

1. *To find the volume of a wedge.*

If the length of the back of a wedge be equal to the length of an edge, the wedge may be considered as a triangular prism, and its volume calculated as in that case. If they are not equal, it may be found by the following rule:

Add twice the length of the base to the length of the edge; multiply the sum by one sixth the product of the breadth of the base and the altitude.



Demonstration.—Let $ABCD$ be the base of a wedge, EF its edge, and FG its altitude. Let $AB=L$, $AD=B$, $FG=h$, and $EF=l$.

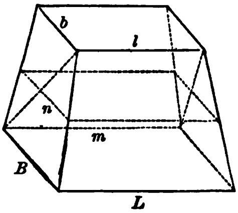
Pass a plane through F, parallel to EAD, cutting the wedge into two portions. One portion, $EAD-K$, is a prism whose volume is $\frac{1}{2}B\times h\times l$. The other is a pyramid, $F-HKCB$, whose volume is $\frac{1}{3}B(L-l)h$. Therefore the whole volume is $\frac{1}{2}Bhl+\frac{1}{3}B(L-l)h=\frac{1}{6}Bh(2L+l)$.

2. *To find the volume of a rectangular prismoid.*

A rectangular prismoid is the frustum of a rectangular pyramid. Its volume may be found by the rule for the frustum of a pyramid, but the following rule is more convenient and capable of extended application:

Add together the areas of the two ends and four times the area of a parallel section midway between them, and multiply the sum by one sixth the altitude.

Demonstration.—Let B and b represent the breadth of the two ends, L and l their lengths, n and m the breadth and length of the middle section, and h the altitude of the prismoid.



Divide it into two wedges by a plane through the opposite edges of the upper and lower base. The volumes of these wedges are

$$\tfrac{1}{6}hB(2L+l) \quad \text{and} \quad \tfrac{1}{6}hb(2l+L).$$

Adding these, we obtain for the volume of the frustum

$$\tfrac{1}{6}h(2BL+Bl+2bl+bL).$$

But $\quad 2m=L+l, \quad$ and $\quad 2N=B+b$;

hence, $\quad 4mn=BL+Bl+bL+bl.$

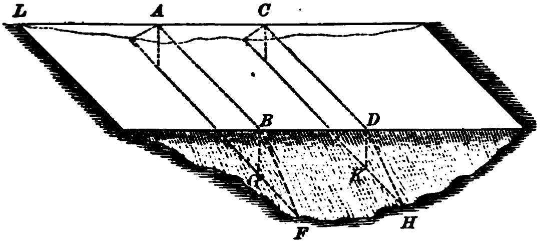
Put $4mn$ in place of its value in the preceding expression, and it becomes

$$\text{Volume of prismoid} = \tfrac{1}{6}h(BL+4mn+bl).$$

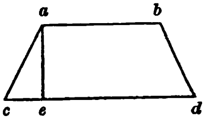
Corollary.—Since the volumes of all frustums of pyramids and cones having equal bases and altitudes are equal, and moreover, since all parallel sections at equal distances from the bases are equal, it follows that the above expression for the volume of a rectangular prismoid is applicable to any prismoid whatever, whether the section be round or polygonal. It may also be shown by higher analysis that it is true for the sphere, spheroid, cone, cylinder, pyramid and many others, with zones and segments of them. It is called the *prismoidal formula*, and has many practical applications.

3. *To calculate the volume of a railroad excavation or embankment.*

Suppose the embankment to be divided, by transverse vertical planes, into solids; these, if the planes be made sufficiently close together, may be regarded as prismoids. The horizontal distance between the cutting planes is the altitude of the prismoid. The bases are trapezoids, for the embankment must be wider at the bottom than the top, and the excavation the



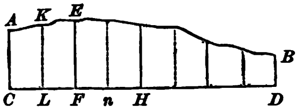
reverse. The depth of the section in the middle of each prismoid is the mean depth of the two end sections, and its breadth is determined by the slope of the embankment. For instance, take the prismoid *CF*. Its altitude is *AC*. The end bases are the trapezoids *AF* and *CH*. The midway section has one side, *ab*, equal to *AB*; the depth, *ae*, equal to $\frac{1}{2}(BG+DK)$, and the other side, *cd*, determined by the slope of *ac* and *bd*. If that slope be 1 vertical to $1\frac{1}{2}$ horizontal, then $ce = 1\frac{1}{2}ae$, and $cd = ab + 3ae$.



The section *LF* has one base equal to zero.

We must apply the prismoidal formula to each prismoid and add the results.

4. Instead of working each prismoid separately, we may, if all the sections be equidistant, develop a formula more simple of application.



Let *AB* represent the surface of the excavation, and *CD* the line of the road. Suppose the full lines to represent the equidistant sections, and the dotted lines the middle sections of the prismoids. Let S_1 represent the section *AC*, S_2 the section KL, and so on, and S_n the last section, BD; also let CL be represented by l.

Then, according to the prismoidal formula,

$$\text{Prismoid } AF = \frac{2l}{6}(S_1 + 4S_2 + S_3),$$

and

$$\text{Prismoid } EH = \frac{2l}{6}(S_3 + 4S_4 + S_5), \text{ etc. etc.}$$

Adding all these, we have for the contents of the whole excavation, the expression

$$\frac{l}{3}(S_1 + 4S_2 + 2S_3 + 4S_4 \ldots . S_n).$$

Hence the following rule:

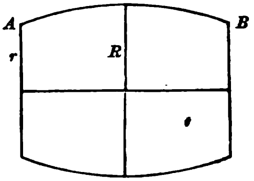
"*Divide the length of the cutting into an even number of equal parts, and find the areas of the transverse sections at the points of division.*

"*Add together the areas of the extreme sections, twice the sum of the areas of all intermediate sections of an odd order, and four times the areas of all intermediate sections of an even order, and multiply by one third the distance between two consecutive sections.*"

On account of the irregularity of the ground, neither of these rules will give precisely correct results. The nearer we take the sections to each other, the more accurate the result will be.

5. GAUGING OF CASKS.—In order to calculate the capacity of a cask, it is necessary to know the following internal dimensions:

$2R$ = the diameter at the bung,
$2r$ = the diameter at the head,
l = the length of the cask.



I. Let AB be an arc of a circle. Apply the prismoidal formula, and the capacity of the cask

$$= V = \frac{l}{6}(\pi r^2 + 4\pi R^2 + \pi r^2)$$

$$= \tfrac{1}{3}\pi l(r^2 + 2R^2).$$

II. It has been found by an examination of a great number of casks that the curvature is confined to the middle third of the length, and is an arc of a parabola, while the end thirds are frustums of cones. The investigation is too abstruse for insertion here, but the following formula has been deduced, and is found to give correct results in most cases:

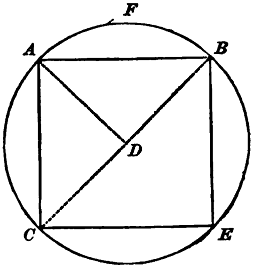
$$V = \frac{\pi l}{90}(39R^2 + 26Rr + 25r^2),$$

or, in case we use diameters, the formula is

$$V' = \frac{\pi l}{360}(39D^2 + 26Dd + 25d^2),$$

D and d being the bung and head diameters. This is known as *Hutton's Rule.*

6. Lumber and Log Measure.—Sawed boards are measured by the square feet of flat surface, supposing the boards to be one inch thick. Thus, a board 12 feet long, 1 feet wide, and 1 inch thick, would contain 12 feet. A board 12 feet long, 1 foot wide, and 2 inches thick, would contain 24 feet. A board 12 feet long, 1½ feet wide, and 1½ inches thick, would contain 27 feet.



It is frequently required to ascertain the number of feet of boards which may be obtained from a given log. The slabs, AFB, etc., are considered worthless; the remaining rectangle, AE, is supposed capable of making boards without any loss from sawing. Thus, if AE were the end of a log 12 feet long, and AB, AC were each 1 foot, the log would be considered as able to make 144 feet of boards.

If AE be a square, its area is equal to the product of CB and AD. This product multiplied by the length will give the cubical contents of the squared log; and this multiplied by 12 will give the feet of boards. It is most convenient to measure the diameter in inches and the length in feet. In this case the result must be divided by 144 to obtain the cubical feet. But as this must be afterward multiplied by 12 to reduce it to board measure, the two operations may be condensed by simply dividing by 12. Hence the following rule:

Multiply the diameter in inches by half the diameter in inches, and the product by the length in feet, and divide by 12. The result will be the feet of boards.

If the log taper, the mean diameter is taken. The bark must be removed before measuring it.

The following rule, which is evidently an approximation, is given as producing the most satisfactory results:

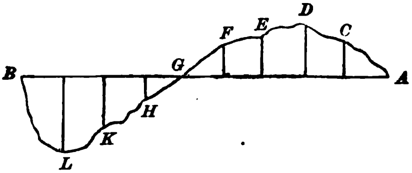
Add together the two extreme diameters in inches, and divide by 2 for the mean diameter. Subtract one third for the side of the square the log will make when hewn. Square the side thus obtained, and multiply the result by the length of the log in feet. Divide by 144. The quotient will be the cubical contents in feet and twelfths of a foot.

This rule gives less than the preceding, and, as trees do not grow exactly round or regular, is probably, as a general rule, more accurate.

EXAMPLES.

1. The base of a wedge is 8 inches by 2, the edge 6 inches, the altitude 10 inches. What is the volume? *Ans.* 73⅓.

2. A level road 10 feet wide is to be laid out from A to B; it is required to know whether the cuttings will fill the embankment, the slope being 1½ horizontal to 1 vertical. The sections were taken 50 feet apart, horizontal measurement, and the levelling showed the following results:



C above A, 20 feet.	G above A, 0 feet.
D above A, 28 feet.	H below A, 16 feet.
E above A, 24 feet.	K below A, 30 feet.
F above A, 18 feet.	L below A, 42 feet.

$$\text{Vol. of cutting } = \frac{l}{3}(S_1+4S_2+2S_3+4S_4+2S_5+S_6)$$

$$= \frac{50}{3}(0+3200+2912+4416+1332+0)$$

$$= 197666.67 \text{ cu. ft.}$$

$$\text{Vol. of embank.} = \frac{l}{3}(S_1+4S_2+2S_3+4S_4+S_5)$$

$$= \frac{50}{3}(0+2176+3300+12264+0)$$

$$= 295666.67 \text{ cu. ft.}$$

3. The staves of a cask are in the form of an arc of a circle. The bung diameter is 3 feet, the head diameter 2½ feet, and the length 4 feet. How many gallons will it contain?

$$V = \tfrac{1}{3}\pi l(r^2+2R^2) = \frac{\pi}{3}4(\tfrac{25}{16}+\tfrac{9}{2}) = 25.3946 \text{ cu. ft.} = 189.95 \text{ gals.}$$

4. What would be the capacity of the cask by *Hutton's Rule?*

$$V = \frac{\pi l}{360}(39D^2+26Dd+25d^2) = \frac{\pi}{90}(351+195+156.25) = 24.5045 \text{ cu. ft.}$$

183.29 gals.

5. How many cubic feet and how many feet of boards in a squared log 20 feet long, which when round measured 24 inches in diameter at one end and 18 at the other, by the two rules?

1st rule, Cu. ft., 30.625; ft. boards, 367.5.
2d rule, Cu. ft., 27.222; ft. boards, 326.664.

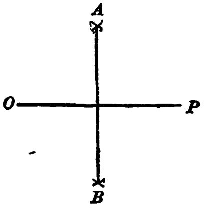
PART III.

MODERN GEOMETRY.

SECTION I.

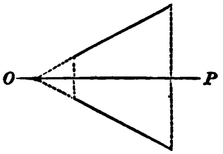
SYMMETRY.

I. *With respect to an axis.*



Two points are **symmetrical** with respect to a line, when the line joining them is bisected at right angles by the given line; thus, A and B are symmetrical with respect to the line OP when AB is bisected at right angles by OP.

The line OP is called the **axis of symmetry**.



Two lines, surfaces or solids are **symmetrical** with respect to a line, when any point in one has a symmetrical point in the other.

The following propositions are easily deduced. The student should prove them:

1. If one of two symmetrical lines intersect the axis of symmetry, the other intersects it in the same point.

2. They make equal angles with it.

3. If two points of a straight line be symmetrical with two points of another, the lines are symmetrical throughout.

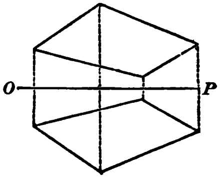
4. If one line be concave or convex to the axis of symmetry, the other is also concave or convex.

5. Any diameter of a circle is an axis of symmetry.

6. A line through the opposite angles of a regular polygon of an even number of sides is an axis of symmetry.

7. A line through an angle, and the middle point of the opposite side of a regular polygon of an odd number of sides is an axis of symmetry.

8. Two symmetrical polygons are equal. Because every point of one has a symmetrical point in the other, either may be revolved about the axis until it coincides with the other; the corresponding symmetrical lines are homologous.



9. Any diameter of a sphere is an axis of symmetry.

10. The axis of a cylinder or cone is an axis of symmetry.

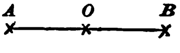
11. A diagonal of a rhomboid is not an axis of symmetry.

Note.—Let the student take various surfaces and solids (such as books and houses), and determine the various axes, centres and planes of symmetry.

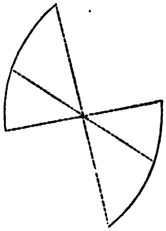
II. *With respect to a centre.*

Two points are **symmetrical** with respect to another point when the line joining them is bisected at this point.

If $AO = BO$, then A and B are symmetrical with respect to O. O is called a **centre of symmetry.**



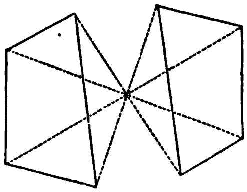
Two lines, surfaces or solids are symmetrical with respect to a centre, when any point of one has a symmetrical point in the other.



The following propositions may be deduced:

1. If one of two symmetrical lines pass through the centre of symmetry, the other will also.

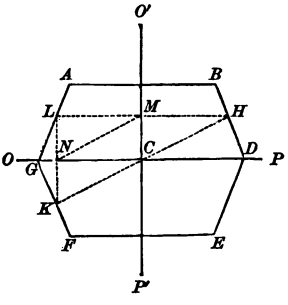
2. If two points of a straight line be symmetrical with two points of another, the lines are symmetrical throughout, and the portions between two pairs of symmetrical points are equal.



3. A centre of a circle or of a regular polygon is a centre of symmetry.

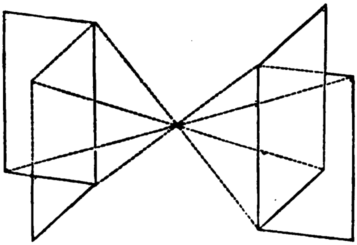
4. Two symmetrical polygons are equal; one of them may be revolved about the centre until it coincides with the other.

5. If a figure be symmetrical with respect to two axes cutting each other at right angles, it is symmetrical with respect to their intersection as a centre.



Let *ABDEFG* be a polygon, symmetrical with respect to *OP*, *O'P'*; it is symmetrical with respect to *C*.

Take any point *H*; join *HC*; produce it to *K*; draw *KL*, *HL* parallel to the axes. The student should fill up the proof of the following steps: *KL* and *HL* will meet in a side of the polygon. The two triangles *HMC*, *CNK* will have *HM*, *MC* and *HMC* equal to *CN*, *NK* and *CNK*; therefore *HC* = *CK*, and the same being true of any points, the polygon is symmetrical with respect to the centre C.



6. Two symmetrical diedral angles are equivalent.

7. Two symmetrical solid angles are equivalent.

8. Two symmetrical polyedrons are equivalent, for their faces are symmetrical, and therefore equivalent.

9. The intersection of the diagonals of a parallelopiped is a centre of symmetry.

10. The centre of a sphere is a centre of symmetry.

11. A cone or pyramid has no centre of symmetry.

III. *With respect to a plane.*

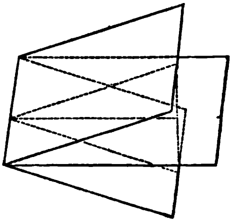
Two points are **symmetrical** with respect to a plane when the plane bisects at right angles the line joining them.

The plane is called the **plane of symmetry.**

Two lines, surfaces or solids are symmetrical with respect to a plane when any point in one has a symmetrical point in the other.

The following propositions may be deduced:

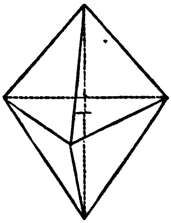
1. Two symmetrical finite lines are equal.



2. If one line intersect the plane of symmetry in a point, the other will in the same point, and make an equal angle with it.

3. Two symmetrical planes make equal angles with the plane of symmetry, and intersect it in the same line.

4. Symmetrical triangular pyramids are equivalent.



Take the plane of the base as the plane of symmetry; hence the two pyramids will have the same base and equal altitudes.

5. Symmetrical polyedrons are equivalent.

Take two symmetrical points within the polyedrons. Decompose the polyedrons into triangular pyramids, of which these symmetrical points are the common vertices. The triangular pyramids will be symmetrical, two and two, and therefore equivalent. Hence the whole polyedrons will be equivalent.

6. Any plane of a great circle is a plane of symmetry of the sphere.

7. There are nine planes of symmetry of the cube.

8. If a figure have two planes of symmetry at right angles to each other, their intersection is an axis of symmetry.

9. If a figure have three planes of symmetry at right angles to each other, their intersection is a centre of symmetry.

10. How many planes of symmetry in the Capitol at Washington?

SECTION II.

LOCI.

A **locus** is the line or surface containing all points having a common property.

The idea of loci may best be obtained by examples:

1. What is the locus of a point in a plane at a given distance from another point?

The common property here is, that all positions of the point are at the same distance from a fixed point. It is evident they must lie in the circumference of a circle. Hence this circumference is the required locus.

2. Find the locus of a point in a plane at a given distance from a straight line. Parallel line.

3. Locus of a point in space at a given distance from another point. Surface of a sphere.

4. Locus of a point in space at a given distance from a straight line. Surface of a cylinder.

5. Locus of a centre of a circle which passes through two given points.

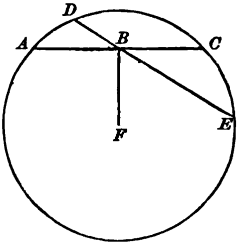
From (III. 1) all circles passing through these points must have their centres in a perpendicular to the line joining the points at its middle point. Hence the required locus is this straight line.

6. Locus of the centre of a circle which is tangent to two given lines.

Line bisecting the angle formed by the given lines.

7. Locus of the centre of a circle passing through a given point and having a given radius. Circle with given point as centre.

8. Locus of a point which divides the chords of a circle in a given ratio.



Let DE be a chord, and let $\frac{DB}{BE}$ be the given ratio. Draw AC so that it will be bisected at B; then (V. 26) $DB \cdot BE = AB^2$. If AC be placed in different positions in the circle, all chords through its middle points will be divided proportionally to the parts of DE. Hence the locus is a circle with a radius FB.

9. Locus of point which is equally distant from two given points.

A plane perpendicular to line joining the points, through its middle point.

10. Locus of point which is equally distant from two given straight lines in same plane. Plane between the lines.

11. Locus of middle point of a line joining two perpendicular but non-intersecting lines. Parallel plane between the lines.

12. Locus of point at a distance a from a point A, and a distance b from point B.

Circle which is the intersection of two spheres.

SECTION III.

MAXIMA AND MINIMA.

DEFINITIONS.

1. Of all quantities of the same kind that which is greatest is the **maximum**, and that which is least is the **minimum**.

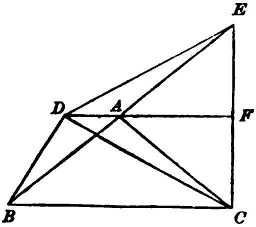
2. Figures which have equal perimeters are said to be **isoperimetric.**

Proposition 1.

***Theorem.**—Of all equal triangles on the same base, that which is isosceles has the minimum perimeter.*

Let ABC be an isosceles triangle, and DBC an equal triangle on the same base; then the perimeter of ABC will be less than the perimeter of DBC.

Produce BA; making $AE = AB = AC$. Join CE; also DA, and produce it to F; join DE. Because the triangles DBC, ABC are equal, DF is parallel to BC.



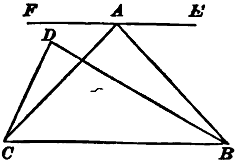
$\therefore EAF = ABC$ and $FAC = ACB$; $\therefore EAF = CAF$; also $AEF = ACF$, and AF is common to the triangles EAF and CAF. Hence $EF = CF$, and the angles at F are right angles. Then in the triangles DEF, DCF DF, FE, and DFE are equal to DF, FC and DFC, each to each; therefore $DE = DC$. Hence $BD + DE = BD + DC$; but $BE < BD + DE$; $\therefore BA + AC < BD + DC$; and this is true no matter in what point of the line DF, D be taken. Hence $BA + AC$ is a minimum.

Corollary.—Of all equal triangles, that which is equilateral has the minimum perimeter.

For it must be isosceles, whichever side is taken for a base.

Proposition 2.

Theorem.—Of all isoperimetric triangles on the same base, that which is isosceles is the maximum.



Let ABC be an isosceles triangle, and let DBC, on the same base BC, have an equal perimeter; then $\triangle ABC > \triangle DBC$.

The point D will fall between the parallels BC and EF.

For, if it fell on EF, the triangles could not be isoperimetric (Prop. I.);* still less could they be so if it fell above EF; hence it falls below EF, and the area BDC is less than the area BAC.

Corollary.—Of all isoperimetric triangles, that which is equilateral is a maximum.

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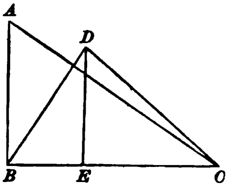
* References such as this are to propositions in Modern Geometry. Others are referred to as heretofore.

Proposition 3.

Theorem.—*Of all triangles having two sides equal, each to each, that which has these sides at right angles is the maximum.*

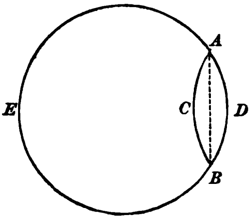


Let AB, BC, at right angles to each other, be equal, each to each, to DB, BC, not at right angles to each other; ABC is greater than DBC.

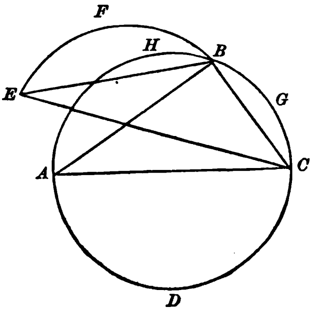
Draw the altitude DE; AB is greater than DE; hence ABC is greater than DBC.

Proposition 4.

Theorem.—*Of all isoperimetric figures, the circle is the maximum.*

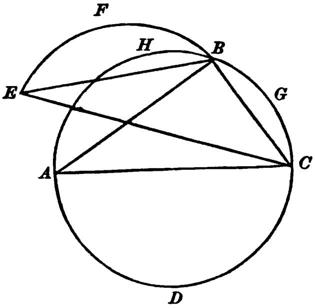


1. The given maximum figure must be convex; that is, a line joining any two points of the perimeter must fall inside the perimeter. For, if $EACB$ be a figure not convex, for the part ACB could be substituted the convex line ADB of equal length, which would increase the area of the figure without increasing the perimeter. Hence, any figure not convex cannot be a maximum.



2. Let $ABCD$ be the maximum figure. Draw AC bisecting the perimeter; it will also bisect the area. For, if not, let ABC be greater than ADC; then a line equal to ABC could be substituted for ADC, which would, with an equal perimeter, cut off a greater area. But $ABCD$ was, by hypothesis, the

maximum. Hence, the area is bisected by *AC*.

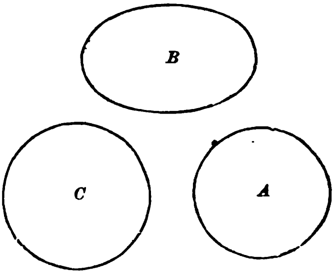


Also, take any point *B* in the perimeter; the angle *ABC* is a right angle.. For, if not, draw *BE* at right angles to *BC*, making it equal to *BA*, and on it describe a segment equal to the segment on *BA*. Then (Prop. 3) the triangle *EBC* is greater than the triangle *ABC*; hence, the whole figure *EFBGC* is greater than the whole figure *AHBGC*; and if symmetrical figures be described on the opposite sides of *EC* and *AC*, the figure on *EC* would be greater than the figure on *AC*. But they have equal perimeters. Hence, *ABCD* is not a maximum, which is contrary to the hypothesis. Therefore, *ABC* is a right angle.

Now, *B* is any point of the perimeter; hence, *ABCD* is a circle, and *AC* its diameter.

Proposition 5.

Theorem.—Of all equal figures, the circle has the minimum perimeter.

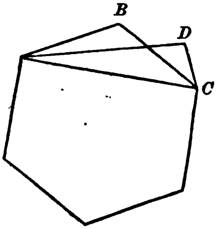


Let *A* be a circle, and *B* any other equal figure; then the perimeter of *A* will be less than the perimeter of *B*.

Let *C* be a circle having the same perimeter as *B*; then (Prop. 4) $C > B$; $\therefore C > A$; hence, the circumference of *C* is greater than the circumference of *A*; therefore, also, the perimeter of *B* is greater than the circumference of *A*.

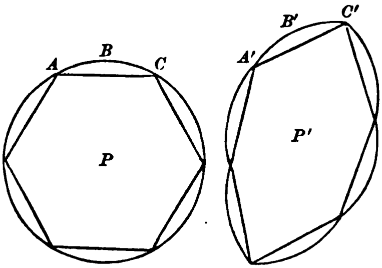
Proposition 6.

Theorem.—Of all isoperimetric polygons having the same number of sides, the regular polygon is the maximum.



1. The maximum polygon must be equilateral; for, if not, an isosceles triangle, *ABC*, could be substituted for *ADC* of equal perimeter, and which would be greater (Prop. 2), and thus the area of the polygon would be increased without changing the perimeter.

2. The maximum polygon must be such as to be inscribed in a circle. For, let *P*, *P′* be equilateral polygons of the same number of sides, of which *P* is inscribed in a circle. On the sides of *P′* describe segments of circles *A′B′C′*, etc., equal to the segments *ABC*, etc. The whole figure *A′B′C′*, etc., thus formed, has the same perimeter as the circle about *P*, and is therefore (Prop. 4) less than the circle. But the segments being equal, *P* must be greater than *P′*.

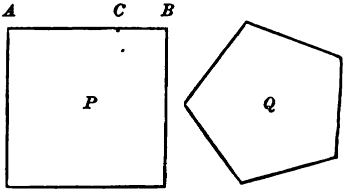


Hence, the maximum polygon, being equilateral and inscriptible, is regular.

Proposition 7.

Theorem.—Of all isoperimetric polygons, that which has the greatest number of sides is the greatest.

Let *P*, *Q* be regular polygons of four and five sides, and having equal perimeters. Take any point C in AB. Then P may be considered as an irregular polygon of five sides, AC, CB being two. Hence (Prop. 6), P is less than Q.



In the same manner, Q may be proved less than an isoperimetric regular polygon of six sides, and so on.

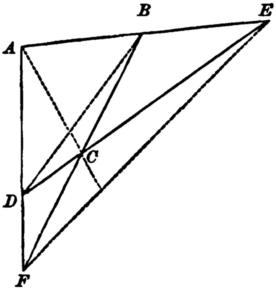
SECTION IV.

TRANSVERSALS.

DEFINITIONS.

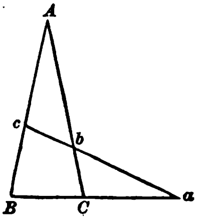
1. A **transversal** is a line cutting a system of lines.

2. A **complete quadrilateral** is the figure formed by four straight lines intersecting in six points, as *ABCDEF*.

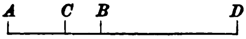


3. The **diagonals** of a complete quadrilateral are the three lines joining the opposite angles—viz. *DB*, *AC*, *EF*.

4. If a transversal cut the three sides of a triangle (produced if necessary), there are formed six segments, of which any two about an angle, as *Ac*, *Ab*, or *Ba*, *Bc*, are said to be **adjacent**, and any other two, as *Ab*, *Ba*, **non-adjacent.**



5. A line is divided **harmonically** when it is divided internally and externally in the same ratio; thus AB is divided harmonically when



$$AC : CB :: AD : DB,$$

or, which is the same, when

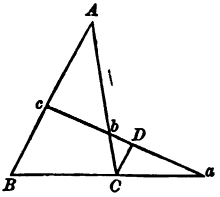
$$AC : AD :: BC : BD.$$

6. C and D are said to be **harmonic conjugates**, as also A and B.

Proposition 8.

Theorem.—*If a transversal cut the three sides of a triangle, the product of three non-adjacent segments is equal to the product of the other three.*

Let the transversal abc cut the three sides of the triangle ABC.



Draw CD parallel to AB.

Then in the similar triangles aCD, aBc,

$$aC : CD :: aB : Bc;$$

also in the similar triangles CDb, Abc,

$$CD : Cb :: Ac : Ab;$$

multiplying together the corresponding terms,

$$aC : Cb :: aB \,.\, Ac : Bc \,.\, Ab;$$

hence, $$aC \,.\, bA \,.\, cB = aB \,.\, bC \,.\, cA.$$

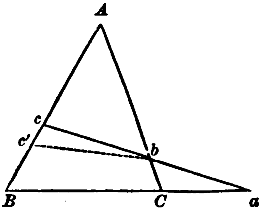
Proposition 9.

Theorem.—*If in the sides of a triangle (produced if necessary) three points be taken, so that the product of three non-adjacent segments be equal to the product of the other three, these points will be in one straight line.*

Let the three points a, b, c be taken in the sides of ABC, so that

$$aC.bA.cB = aB.bC.cA;$$

then will abc be a straight line.



For if not, let abc' be a straight line, then (Prop. 8)

$$aC.bA.c'B = aB.bC.c'A.$$

But (Hyp.),

$$aC.bA.cB = aB.bC.cA,$$

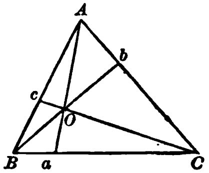
whence, by division $\frac{c'B}{cB} = \frac{c'A}{cA}$; but this can only be true when c' coincides with c; hence abc is a straight line.

Proposition 10.

***Theorem.**—The three lines drawn from any point in the plane of a triangle to the three angles, and cutting the opposite sides (produced if necessary), will cut them so that the product of three non-adjacent segments is equal to the product of three others.*

Let the three lines, AOa, BOb, COc, be drawn through the angles of ABC and any point, O, in its plane,

then $\quad aB.bC.cA = aC.bA.cB.$



Because the triangle ACa is cut by the transversal Bb,

(Prop. 8), $\quad Ba.OA.bC = BC.Oa.bA.$

Also because the triangle ABa is cut by the transversal Cc,

$$CB.Oa.cA = Ca.OA.cB.$$

Multiplying these together, we have

$$aB.bC.cA = aC.bA.cB.$$

Proposition 11.

Theorem.—*If three points be taken on the sides of a triangle, so that the product of three non-adjacent segments is equal to the product of the other three, the three straight lines joining these points with the opposite angles will pass through the same point.*

The proof is similar to (Prop. 9).

From the last theorem the following propositions may be proved:

1. The three lines drawn from the angles of a triangle to the middle points of the opposite sides intersect in the same point.

2. The three lines drawn bisecting the angles of a triangle intersect in the same point.

3. The three perpendiculars drawn from the angles of a triangle to the opposite sides intersect in the same point.

Proposition 12.

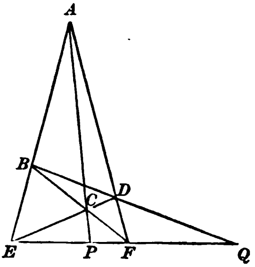
Theorem.—*In a complete quadrilateral each diagonal is divided harmonically by the other two.*

Let the diagonal EF be cut by the diagonals AC, BD in P and Q; then will

$$PE : PF :: QE : QF.$$

Because the transversal QDB cuts the triangle AEF, we have (Prop. 8)

$$QE.DF.BA = QF.DA.BE.$$



Because the straight lines CA, CF, CE cut the opposite sides of the triangle AEF, we have (Prop. 10)

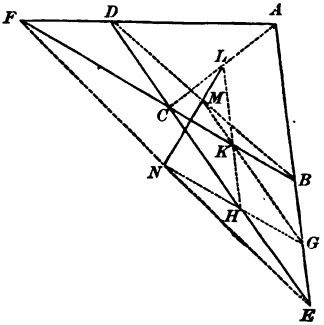
$$PE.DF.BA = PF.DA.BE.$$

Dividing one of these equations by the other, $\frac{PE}{QE} = \frac{PF}{QF}$, whence $PE : PF :: QE : QF$; hence EF is divided harmonically in P and Q.

The same may be proved of either of the diagonals AC, BD.

Proposition 13.

***Theorem.**—The middle points of the three diagonals of a complete quadrilateral are in the same line.*



Let L, M, N be the three middle points of the diagonals AC, BD, EF. LMN is a straight line.

Bisect the sides of the triangle BCE in G, H and K. The line HK will be parallel to AE, and will meet AC in L; similarly, GH will pass through N, and GK through M.

Considering FDA as a transversal of the triangle EBC (Prop. 8), gives

$$EA \,.\, BF \,.\, CD = ED \,.\, BA \,.\, CF.$$

Taking the half of each of these lines,

$$HL \,.\, GN \,.\, MK = GM \,.\, KL \,.\, HN.$$

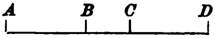
Hence, LMN may be considered a transversal of the triangle GHK, and is a straight line (Prop. 9).

SECTION V.

ANHARMONIC AND HARMONIC RATIO.

DEFINITIONS.

1. THE **anharmonic ratio** of four points is the quotient obtained by dividing the ratio of the distances of the first two from the third, by the ratio of the distances of the first two from the fourth.



Thus, the anharmonic ratio of the four points A, B, C, D is

$$\frac{AC}{BC} \div \frac{AD}{BD},$$

and is denoted for brevity by $[ABCD]$.

The anharmonic ratio $[BCAD]$ would be $\frac{BA}{CA} \div \frac{BD}{CD}$.

2. If the anharmonic ratio of four points be equal to unity, they are said to be in **harmonic ratio.**

Thus, if $\quad \frac{AC}{BC} \div \frac{AD}{BD} = 1, \quad$ or $\quad \frac{AC}{BC} = \frac{AD}{BD},$

A, B, C, D are harmonic points, which agrees with Def. 5, page 245.

3. A **pencil** is a system of straight lines diverging from a point.

4. A **ray** is one of these diverging lines.

5. The **vertex** of the pencil is the point from which they diverge.

Proposition 14.

Theorem.—*The anharmonic ratio of a system of points is not changed by interchanging two of the points, provided the other two points be also interchanged.*

For $$[ABCD]=\frac{AC}{BC}\div\frac{AD}{BD}=\frac{AC.BD}{BC.AD};$$

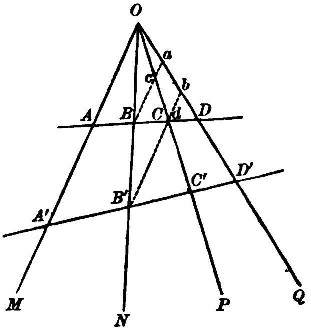
also, $$[BADC]=\frac{BD}{AD}\div\frac{BC}{AC}=\frac{AC.BD}{BC.AD}.$$

Hence, also, $[ABCD]=[BADC]=[CDAB]=[DCBA]$.

Scholium.—There are twenty-four ways in which these four letters may be arranged; hence there are six different anharmonic ratios which may be formed from them.

Proposition 15.

Theorem.—*If a pencil of four rays be cut by any two transversals, the anharmonic ratio of the four cutting points of one transversal is equal to the anharmonic ratio of the four corresponding points of the second.*



Let OM, ON, OP, OQ be four rays, and $ABCD$, $A'B'C'D'$ any two transversals; then will $[ABCD]=[A'B'C'D']$.

From B and B' draw Ba, $B'b$ parallel to AO, cutting OC' in c and d.

From similar triangles CAO, CBc, and also DAO, DBa,

$$CA : CB :: OA : cB,$$

and $$DA : DB :: OA : aB.$$

Dividing one proportion by the other,

$$\frac{CA}{CB} \div \frac{DA}{DB}, \quad \text{or} \quad [ABCD] = \frac{cB}{aB}.$$

Similarly, $$[A'B'C'D'] = \frac{dB'}{bB'}.$$

But $$\frac{cB}{aB} = \frac{dB'}{bB'};$$

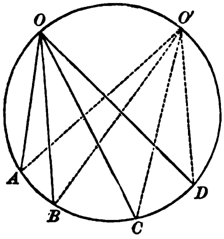
hence, $$[ABCD] = [A'B'C'D'].$$

Scholium.—The anharmonic ratio of a pencil of four rays is the anharmonic ratio of four points cut from the rays by any transversal. The angles of a pencil are the six angles formed by the four rays with each other, taken two and two. Hence, *the values of the anharmonic ratios of two pencils are equal, when the rays make equal angles with each other, each to each.*

The anharmonic ratio of a pencil is represented by $[O-ABCD]$.

Proposition 16.

Theorem.—*If four points be taken in the circumference of a circle, and lines be drawn to any other point in the circumference, the anharmonic ratio of the pencil is the same, whatever the position of the fifth point.*



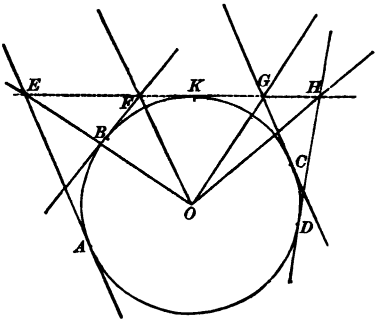
For the angles at O and O' are equal; hence (Prop. 15, Sch.),

$$[O\text{-}ABCD] = [O'\text{-}ABCD].$$

Proposition 17.

Theorem.—*If the centre of a circle be joined with the points of intersection of four fixed tangents with a fifth tangent, the anharmonic ratio of the pencil formed is the same, whatever be the position of the fifth tangent.*

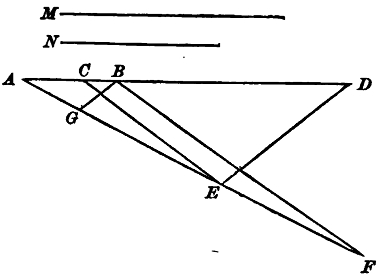
Let four tangents touch the circle in A, B, C, D, and intersect a fifth tangent in E, F, G, H; then will $[O\text{-}EFGH]$ be the same, whatever the position of $EFGH$.



For the angle EOK is one half AOK, and FOK is one half BOK; therefore EOF is one half the fixed angle AOB, and is therefore constant, whatever the position of $EFGH$. The same may be proved of FOG and GOH. Hence the anharmonic ratio of the pencil is the same, whatever the position of $EFGH$.

Proposition 18.

Problem.—*To divide a straight line harmonically in a given ratio.*



Let AB be the line and $M:N$ the given ratio; it is required to divide AB harmonically in the ratio $M:N$.

Lay off in any direction $AE = M$ and $EF = EG = N$. Join GB, FB, and draw ED, EC parallel to them.

It may be proven that

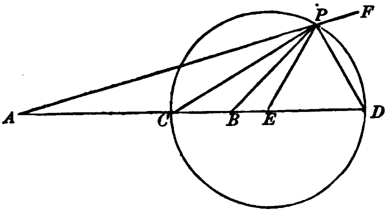
$$AC : BC :: AD : BD :: M : N.$$

Scholium.—If any point not in AB be taken, and a pencil of four rays drawn through A, C, B, D, it is said to be a *harmonic pencil.*

Proposition 19.

Problem.—*To find the locus of all points whose distances from two given points are in a given ratio.*

Let A and B be the points, and let $M : N$ be the given ratio.



Divide (Prop. 18) AB harmonically in C and D; on CD describe a circle. The circumference of this circle will be the required locus.

Take any point, P, and join PA, PC, PB, PE, and PD. Because AB is divided harmonically in C and D,

$$AD : AC :: BD : BC,$$

or, $$AE + EC : AE - EC :: CE + EB : CE - EB;$$

hence, $$AE : EC :: EC : EB,$$

or, $$AE : EP :: EP : EB.$$

Therefore the sides about the common angle E of the triangles AEP, BEP are proportional; hence the triangles are similar, and the angle BPE is equal to A.

Also $ECP = CAP + CPA$, and its equal

$EPC = EPB + BPC.$

But $EPB = CAP$; $\therefore APC = BPC$,

and (V. 3), $AP : PB :: AC : CB :: M : N.$

But P is any point of the circumference; hence the circumference is the required locus.

Corollary 1.—Because CPD is a right angle, and PC bisects the angle APB, PD must also bisect the angle BPF.

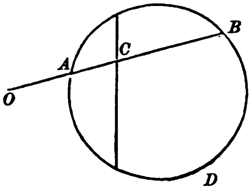
Corollary 2.—If the angles APB, BPF be bisected, then AB is divided harmonically in C and D; and, conversely, if AB be divided harmonically in C and D, the angles APB, BPF will be bisected by PC and PD.

SECTION VI.

POLE AND POLAR IN THE CIRCLE.

DEFINITIONS.

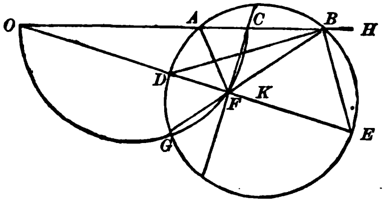
IF from a point, O, without the circle ABD, the secant OAB be drawn, and C, the harmonic conjugate of O with respect to A and B, be taken, then the locus of C, as OAB revolves about O, is called the **polar** of the point O; and O is the **pole** of this locus.

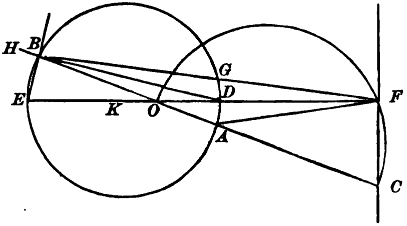


Proposition 20.

***Theorem.**—The polar of a point is a straight line at right angles to the diameter through the point.*

Let O and C be harmonic conjugates with respect to A and B on the circumference ABD. Draw CF at right angles to the diameter DE; and on OC describe a semicircle passing through F. Join BF, and let it cut the circle in G; join AF, BD, BE.





Because AB is divided harmonically in O and C, and F is a point of the circumference OFC, OF bisects (Prop. 19, Cor. 2) AFG; and DE is a diameter; hence the arcs AD, DG are equal, and consequently the angles OBD, DBF; hence, also BE, drawn from B in the circumference DBE, bisects FBH, and therefore OF is divided (Prop. 19, Cor. 2) harmonically in D and E, and F is the harmonic conjugate of O.

Hence F is a fixed point, and C is always in the perpendicular to OE through F; therefore CF is the polar of the pole O.

***Corollary 1.**—The radius of the circle is a mean proportional between the distances of the centre from the pole and the polar.*

For, because $OE : OD :: FE : FD$,
as in last Prop., $OK : DK :: DK : FK$,
or, $OK \cdot FK = DK^2 = R^2$.

***Corollary 2.**—The pole and polar are on the same side of the centre; if the pole be exterior, the polar is interior to the circle, and vice versâ.*

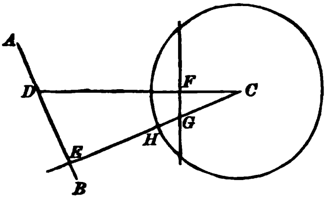
***Corollary 3.**—The polar passes through the point of contact of the tangent through the pole.*

***Corollary 4.**—The polar of the centre is at infinity, and the polar of a point at an infinite distance passes through the centre.*

***Corollary 5.**—The polar of a point on the circle is a tangent through this point.*

Proposition 21.

***Theorem.**—The polars of all points of a straight line pass through the pole of that line; and the poles of all straight lines which pass through a fixed point are in the polar of that point.*

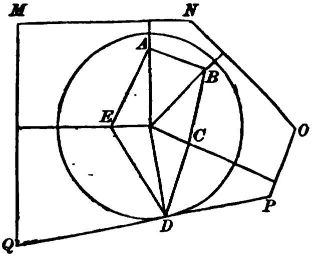


Let AB be a straight line, G its pole with respect to the circle C, and D any point in it. CGE is perpendicular to AB (Prop. 20). Also (Prop. 20, Cor. 1), $CE \cdot CG = CH^2$. Draw FG perpendicular to CD. From the similar triangles DCE, FCG,

$$CE \cdot CG = CD \cdot CF.$$

Hence $CD \cdot CF = CH^2$ and FG is (Prop. 20, Cor. 1) the polar of D.

Hence the polar of any point, D, of the line AB passes through G, and the pole, G, of any line, AB, which passes through D is in the polar FG of the point D.



Scholium.—From this it results that if the angles of the polygon $ABCDE$ be the poles of the sides of the polygon $MNOPQ$, the angles of the second polygon are the poles of the sides of the first. Each polygon is said to be the **reciprocal polar** of the other with respect to the circle, which is called the **directing circle.**

If one polygon be inscribed in the circle, the other will be circumscribed about it, the sides of the latter passing through the angles of the former.

The angles of either may be determined from the sides of the other, or the sides of either from the angles of the other.

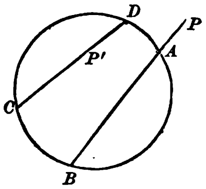
A double set of theorems may be deduced, in which the properties relating to the lines of one may be converted into properties relating to the points of the other, and *vice versâ.* These theorems belong principally to curves of which Elementary Geometry does not treat, and will not be considered here.

SECTION VII.

RADICAL AXES.

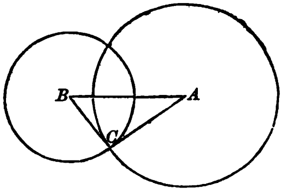
DEFINITIONS.

1. THE **power** of a point with reference to a circle is the rectangle of the segments of a secant through the point, formed by the circumference of the circle.



Thus the power of P with reference to $ABCD$ is $PA \cdot PB$; of P', is $P'C \cdot P'D$.

2. Two circles cut each other **orthogonally** when the radii to the point of intersection are at right angles to each other.

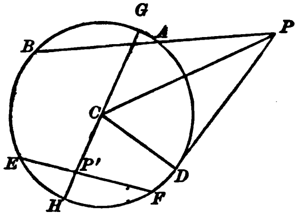


That is, when ACB is a right angle.

3. The **radical axis** of two circles is the locus of all points whose powers, with reference to the circles, are equal.

Proposition 22.

Theorem.*—The power of a point with reference to a circle is equal to the difference of the square of the distance of the point from the centre of the circle, and the square of the radius.*



1. If the point be external,

$AP \cdot PB = (\text{V. } 27)$
$PD^2 = PC^2 - CD^2.$

2. If the point be internal,

$EP' \cdot P'F = (\text{V. } 26)\ GP' \cdot P'H$
$= (CH + CP')(CH - CP')$
$= CH^2 - CP'^2.$

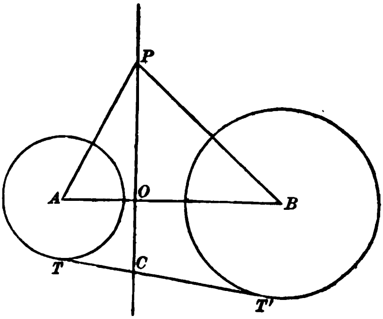
If the point be on the circumference, its power is *zero*.

Corollary 1.*—If the point be external, its power is equal to the square of the tangent from it to the circumference.*

Corollary 2.*—If two circles cut each other orthogonally, the square of either radius is equal to the power of its centre with reference to the other circle.*

Proposition 23.

Theorem.*—The radical axis of two circles is perpendicular to the line joining their centres.*



Because the power of any point P with reference to both circles is the same (Prop. 22),

$$PA^2 - r^2 = PB^2 - r'^2,$$

if r, r' be the radii; or,

$$PB^2 - PA^2 = r'^2 - r^2.$$

Now, draw PO perpendicular to AB, and (I. 42), $PB^2 - PA^2 = OB^2 - OA^2 = r'^2 - r^2.$

Hence, O is a fixed point, and the radical axis is perpendicular to AB through it.

***Corollary* 1.**—*If two circles intersect or touch, their common chord or tangent is a radical axis.*

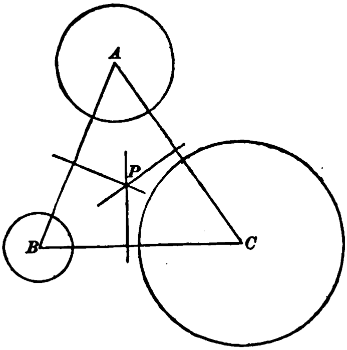
***Corollary* 2.**—*The tangents drawn from any point in the radical axis to the two circles are equal to each other.*

Hence, if TT' be a common tangent, $CT = CT'$; and the radical axis may be found by bisecting two common tangents and joining the points of bisection.

***Corollary* 3.**—*The radical axis of two equal circles passes through the middle of the line through their centres.*

Proposition 24.

Theorem.—*The radical axes of three circles, taken two at a time, pass through the same point.*



Let A, B and C be three circles; the radical axes of A and B and of B and C will meet in some point P.

Now, because P is in the radical axis of A and B, the powers of P with reference to A and B are equal; hence, also, the powers of P with reference to B and C are equal; therefore the powers of P with reference to A and C are equal, and P lies in the radical axis of A and C. Hence, the three radical axes intersect in the point P.

Definition.—The point P is called the **radical centre** of the three circles.

Scholium 1.—If the centres of A, B, C be in a straight line, the radical centre is at an infinite distance.

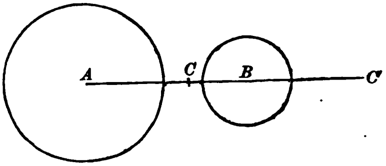
Scholium 2.—This affords an easy method of finding the radical axis of two circles. Draw a third circle cutting the two, and find the intersection of two common chords. This will be a point (Prop. 23, Cor. 1) in the radical axis, which may be drawn perpendicular to the line through the centres.

SECTION VIII.

CENTRES OF SIMILITUDE.

DEFINITIONS.

1. The **centres of similitude** of two circles are the points in which the line joining their centres is divided in the ratio of the radii.



Thus, if R and R' represent the radii of the circles A and B, if

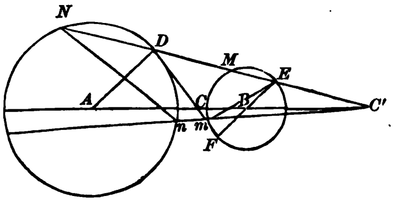
$$AC : BC :: AC' : BC' :: R : R',$$

then C is the **internal**, and C' the **external** centre of similitude.

Corollary.—If the two circles touch each other, the internal centre of similitude is in the point of contact.

Proposition 25.

Theorem.—A line through the extremities of two parallel radii of two circles passes through a centre of similitude.



Let A and B be two circles, and AD, BF and AD, BE be parallel radii; if DF, DE be joined, they will cut AB in C and C', the internal and external centres of similitude.

For $AC : BC :: AD : BF :: R : R'$,

and $AC' : BC' :: AD : BE :: R : R'$.

Hence, C and C' are the centres of similitude.

Corollary 1.*—Any transversal through the centre of similitude is divided in the ratio of the radii.*

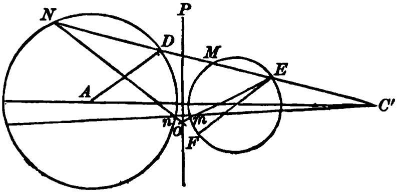
Corollary 2.*—Any tangent through the centre of similitude is divided in the ratio of the radii.*

Corollary 3.*—The line joining the centres is divided harmonically at the centres of similitude.*

Definitions.—The points E and D, as also M and N, are said to be **homologous** with respect to each other; and E and N, as also M and D, are said to be **anti-homologous** with respect to each other. If $C'N$, $C'n$ be two secants, then Em, Nn are said to be **anti-homologous chords.**

Proposition 26.

Theorem.*—The product of the distances of a centre of similitude of two circles, from two anti-homologous points, is constant.*



Let C' be a centre of similitude of the circles A and B; then $C'E \,.\, C'N$ is constant.

Let the power of C' with reference to the circle A, or $C'D . C'N = P$. Also (Prop. 25, Cor. 1), $\frac{C'N}{C'M} = \frac{R}{R'}$. Dividing the first by the second, $C'D . C'M = P\frac{R'}{R}$.

Now, D and M are anti-homologous points, and the second member of the equation is constant. Hence the theorem.

***Corollary 1.**—The anti-homologous chords meet on the radical axis.*

Let Em, Nn be anti-homologous chords meeting at O. Then, from the above,

$$C'E . C'N = C'm . C'n;$$

hence (V. 27, Cor. 3), E, m, n and N are on the circumference of the same circle, and Em is a common chord of this new circle and the circle B; hence (Prop. 23, Cor. 1) it is a radical axis of these two circles. For the same reason Nn is a radical axis of the new circle and the circle A. Hence, O is the radical centre of the three circles, and OP (Prop. 24) the radical axis of A and B.

***Corollary 2.**—The extreme positions of the chords (that is, when they become tangents) also intersect on the radical axis.*

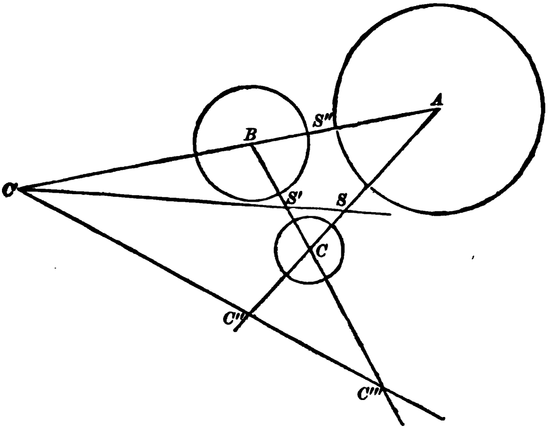
Proposition 27.

***Theorem.**—If three circles be given, and considered, taken two at a time, as forming three pairs—*

1. *The external centres of similitude of the three pairs are in a straight line.*

2. *The external centre of similitude of any pair, and the internal centre of similitude of the other two pairs, are in a straight line.*

Let A, B and C be three circles, C', C'', C''', their external centres of similitude, taken two and two; S and S', two internal centres of similitude; then will $C'C''C'''$ be a straight line, as also S, S', C'.



1. From (Sec. VIII., Def. 1), calling R, R', R'' the radii of A, B and C respectively,

$$\frac{C'A}{C'B}=\frac{R}{R'},\qquad \frac{C''C}{C''A}=\frac{R''}{R},\qquad \frac{C'''B}{C'''C}=\frac{R'}{R''}.$$

Multiplying these together,

$$C'A\,.\,C''C\,.\,C'''B=C'B\,.\,C''A\,.\,C'''C.$$

But $C'C''C'''$ is a transversal cutting, the sides produced of the triangle ABC; hence (Prop. 9), it is a straight line.

2. Also,

$$\frac{SA}{SC}=\frac{R}{R''},\qquad \frac{S'C}{S'B}=\frac{R''}{R'},\qquad \frac{C'B}{C'A}=\frac{R'}{R}.$$

Hence, $$SA\,.\,S'C\,.\,C'B=SC\,.\,S'B\,.\,C'A,$$

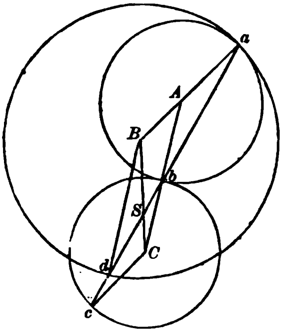
and (Prop. 9) S, S' and C' are in a straight line.

In the same way, S'', S, C''' is a straight line, and also $S''S'C''$.

Definitions.—The line $C'C''C'''$ is called the **external axis of similitude**; and C', S', S is called the **internal axis of similitude.**

Proposition 28.

Theorem.—If a circle touch two other circles, the chord of contact will pass through a centre of similitude.



Let the circle A touch the circles B and C in a and b. The chord ab will pass through S, a centre of similitude of B and C.

The lines AC, BA will pass through the points of contact, b and a. Produce ab to c; join Cc, Bd. Then, because the isosceles triangles Bda, Ccb, Aba are similar, the radii Ba, Cc are parallel. Therefore the chord ab passes through (Prop. 25) a centre of similitude S.

Proposition 29.

Problem.—To describe a circle tangent to three given circles.

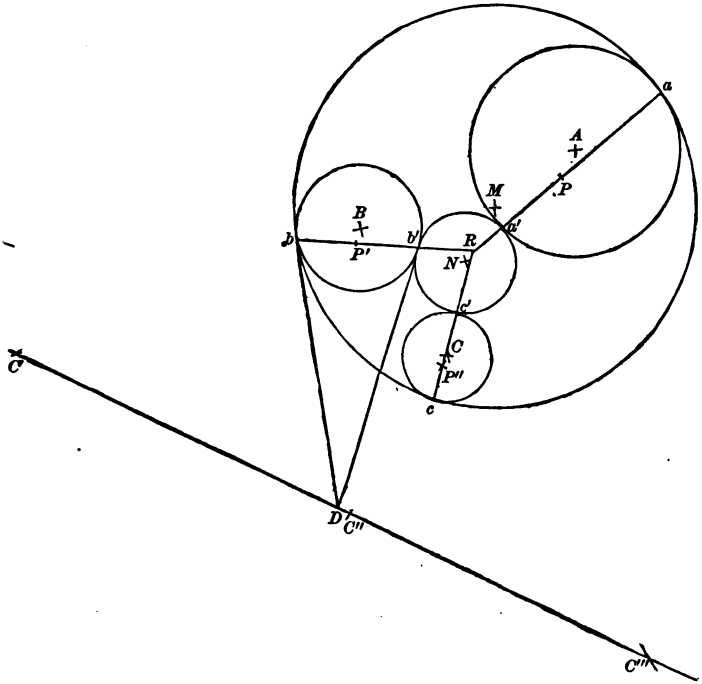
There may be eight circles tangent to the three circles—one touching all three externally, one all three internally; and three other pairs, one of each having internal contact with one circle, and external with two; and the other having internal contact with two, and external with one.

Let A, B, C be the three given circles, $C'C''C'''$ the external axis of similitude.

Suppose the problem solved, and let aa', bb', cc' be the three chords of contact of the circles M and N.

We will prove—

1. That these chords pass through the radical centre of the three given circles.



For, a being an internal centre of similitude (Sec. VIII., Def. I., Cor.) of the circles A and M, and a' of A and N, the chord aa' will pass through (Prop. 28) the external centre of similitude of the circles M and N. For the same reasons will also bb' and cc'. Hence, they meet in R. Also (Prop. 28), ab, $a'b'$ will pass through the external centre of similitude C' of the circles A and B. Hence, aa', bb' are anti-homologous chords (Prop. 25, Def.), and their intersection R is (Prop. 26, Cor. 1) a point in the radical axis of A and B. For the same reasons it is in the radical axis of B and C, and of A and C, and is therefore the radical centre of the three circles.

2. Each of the chords aa', bb', cc' contains the pole, with reference to its own circle, of the external axis of similitude $C'C''C'''$.

For, since R is in the chord of contact of the circle A with the circles M and N, it is (Prop. 28) a centre of similitude of M and N. Hence the two chords ab, $a'b'$ are anti-homologous, and their point of meeting, C', is (Prop. 26, Cor. 1) a point of the radical axis of the circles M and N.

Similarly, it could be shown that this radical axis passes through C'' and C''', and therefore agrees with the external axis of similitude. Hence the line $C'C''C'''$ contains (Prop. 26, Cor. 2) D, the point of meeting of the two anti-homologous tangents Db, Db'; that is, D is the pole (Prop. 20, Cor. 3) of the chord bb' with reference to the circle B; hence (Prop. 21) the chord bb' contains the pole of the polar $C'C''C'''$ with reference to the circle B. Also aa', cc' contain the poles of $C'C''C'''$ with reference to A and C.

Consequently, we have the following rule to solve our problem:

Determine the radical centre, R, and the external axis of similitude of the three circles, A, B, C; take, with reference to each of them, the poles P, P', P'' of this axis, and draw the lines RP, RP', RP'', cutting the circles respectively in a and a', b and b', c and c'; through the points a, b, c pass a circle; it will be the circle of internal contact; through $a'b'c'$ pass a circle; it will be the circle of external contact.

By substituting for the external axis of similitude the internal axes, the other tangent circles may be found in a similar way.